What is a limit?

Suppose you have the following function:

$$f(x) = \frac{x^2 + x - 2}{x - 1} \tag{1}$$

The above function is well-defined except when x = 1 since the denominator will be 0 in such a case. Given this, let's calculate the above function when x is not equal to 1, but slightly bigger than 1. Let's plug in x = 1.01, then we get f(1.01) = 3.01. Similarly,

$$f(1.001) = 3.001, \ f(1.0001) = 3.0001, \ f(1.00003) = 3.00003$$
 (2)

Now let's calculate the above function when x is not equal to 1, but slightly smaller than 1. We get

$$f(0.999) = 2.999, \ f(0.9999) = 2.9999, \ f(0.99999) = 2.99999$$
 (3)

In both cases, we see that the function f approaches 3, when x approaches 1. This is the concept of limit. We can write this relation as follows:

$$\lim_{x \to 1} f(x) = 3 \tag{4}$$

In other words, f(1) is not defined, but the limit is defined. Given this, could we obtain the above formula, without using the calculator? The answer is yes. We can re-write the above equation as follows:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \frac{(x - 1)(x + 2)}{x - 1} = x + 2 = 3$$
(5)

where we have canceled out the common factor (x-1) in the numerator and the denominator. This is possible as long as x is not equal to 1. In our case, this is justified since we are approaching the limit x = 1, but never x = 1.

In limit, the x value one approaches do not need to be a finite number as in our example, but can be infinity or negative infinity. For example,

$$\lim_{x \to \infty} x^2 = \infty \tag{6}$$

(The symbol ∞ denotes infinity.) This is obvious. If you plug in a very big number for x, its square will be a very big number. Another example:

$$\lim_{n \to \infty} \frac{1}{n} = 0 \tag{7}$$

In other words, if you divide 1 by a very big number, it will be a very small number.

Now consider the following example which we will encounter in our article "Bohr model."

$$\lim_{n \to \infty} \frac{n(n-1/2)}{(n-1)^2} = \lim_{n \to \infty} \frac{(1-1/2n)}{(1-1/n)^2} = \frac{1-0/2}{(1-0)^2} = 1$$
(8)

where we have divided the numerator and the denominator by n^2 in the first step, and used (7) in the second step.

Now, let's consider somewhat non-trivial example:

$$\lim_{x \to \infty} \sqrt{x^2 + 2x} - x \tag{9}$$

when x is large, it will be a form of a very big number subtracted by a very big number. Therefore, it is not easy to determine what the answer will be since we do not know how "big" they are. To guess the answer, let's plug in very big numbers. We have:

 $\sqrt{10000^2 + 2 \cdot 10000} - 10000 \approx 10000.99995 - 10000 = 0.99995$ $\sqrt{100000^2 + 2 \cdot 100000} - 100000 \approx 100000.999995 - 100000 = 0.999995$ Therefore, we suspect that (9) is 1. Let's actually prove this:

$$\sqrt{x^{2} + 2x} - x = \frac{(\sqrt{x^{2} + 2x} - x)(\sqrt{x^{2} + 2x} + x)}{(\sqrt{x^{2} + 2x} + x)}$$
$$= \frac{x^{2} + 2x - x^{2}}{\sqrt{x^{2} + 2x} + x} = \frac{2x}{\sqrt{x^{2} + 2x} + x}$$
$$= \frac{2}{\sqrt{1 + 2/x} + 1}$$
(10)

$$\lim_{x \to \infty} \sqrt{x^2 + 2x} - x = \lim_{x \to \infty} \frac{2}{\sqrt{1 + 2/x} + 1} = \frac{2}{1+1} = 1$$
(11)

There is an actually easier way to solve this problem, if you have read our earlier article "The imagination in mathematics: Pascals triangle, combination, and the Taylor series for square root." Note that if 2/x is close to 0, which is the case for big x, (9) is approximately equal to

$$x\sqrt{1+2/x} - x = x(1+2/x)^{1/2} - x \approx x(1+1/x) - x = 1$$
(12)

Of course, if you haven't learned the Taylor series for square root at school, you may not be able to use this method for the test, but it is still useful for multiple choice questions.

Limit is necessary to understand calculus, which is crucial in physics. Limit on its own is also important in physics.

Remark. In this article, we have considered the case in which a function is not defined at a particular point, but the limit at that point is defined. However, limit of a function at a particular point can be defined when the function is well defined at the point as well. For example, if we let g(x) = x + 4

$$\lim_{x \to 3} g(x) = 7 \tag{13}$$

as

$$g(3.001) = 7.001, \quad g(3.00001) = 7.00001, \quad g(3.000002) = 7.000002$$

 $g(2.999) = 6.999, \quad g(2.99999) = 6.99999, \quad g(2.999999) = 6.999999$

show that the value approaches 7 when x approaches 3. Of course, in this case, we have:

$$\lim_{x \to 3} g(x) = g(3) \tag{14}$$

In other words, the limit of function g as x approaches 3 is simply given by the value of the function when x is 3. In the next article, we will talk more about such cases. i.e. cases in which the limit of a function as x approaches a certain number is simply given by the value of the function when x is given by the number it approaches. Certainly, this is a property that many functions satisfy. We will give a name to this property in the next article.

Problem 1. Evaluate the followings. $(Hint^1)$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = ?, \qquad \lim_{x \to 1} \frac{x^3 - 1}{x - 1} = ? \tag{15}$$

Problem 2. Evaluate the followings. $(Hint^2)$

$$\lim_{x \to 0} \frac{x^2}{x^4} = ?, \qquad \lim_{x \to \infty} \frac{3x+5}{\sqrt{x+2}(x-2)} = ?$$
(16)

Problem 3. Evaluate the followings. (Hint³)

$$\lim_{x \to 1} \frac{x^2 + 1}{x + 1} = ?, \qquad \lim_{x \to \infty} \frac{3x + 4}{\sqrt{x + 2\sqrt{2x + 3}}} = ? \tag{17}$$

¹For the first one, use $(x^2 - 4) = (x - 2)(x + 2)$. For the second one, $(x^3 - 1) = (x - 1)(x^2 + x + 1)$.

²For the second one, divide the numerator and the denominator by x.

³For the second one, use the fact that the denominator is equal to $\sqrt{x}\sqrt{1+2/x}\sqrt{x}\sqrt{2+3/x} = x\sqrt{1+2/x}\sqrt{2+3/x}$.

Problem 4. Evaluate the following. $(Hint^4)$

$$\lim_{x \to \infty} \frac{x^2 + 3x - 1}{x^3 - 3x^2 + 1} = ?$$
(18)

Problem 5. Evaluate the followings. (Hint⁵)

$$\lim_{x \to -\infty} x^5 + 100x^4 - 3x = ?, \qquad \lim_{x \to -\infty} x^4 + 30x^3 = ?$$
(19)

Problem 6. Evaluate the following. (Hint⁶)

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 1} = ? \tag{20}$$

Problem 7. Evaluate the followings. $(Hint^7)$

$$\lim_{x \to \infty} 2^x = ?, \qquad \lim_{x \to \infty} 2^{-x} = ?, \qquad \lim_{x \to -\infty} 2^x = ?$$
(21)

Problem 8. Evaluate the following. Here, 0^+ denotes that the limit is approached from values slightly bigger than 0. This is necessary as log of a negative number is not defined. However, as we will see in "Euler's formula and hyperbolic functions" log of a negative number can be defined in complex numbers. (Hint⁸)

$$\lim_{x \to 0^+} \log_2 x =? \tag{22}$$

Summary

• A function f(x) may not defined at certain value $x = x_0$, but its limit $\lim_{x\to x_0} f(x)$ may exist.

⁴The numerator is equal to $x^{2}(1+3/x-1/x^{2})$ and the denominator is equal to $x^{3}(1-x^{2})$ $3/x + 1/x^3$).

⁵Our earlier article "Asymptotic behavior of polynomials" is helpful. ⁶Use $(x^2 - 3x + 2) = (x - 1)(x - 2)$ and $(x^2 - 1) = (x - 1)(x + 1)$.

⁷Use $2^{-x} = 1/(2^x)$ for the second equation.

⁸The answer to Problem 7 can be helpful.