Linear approximations

Jordanna drived from Seoul to Busan by highway non-stop. She passed Daejeon, which is 150 km from Seoul along the highway 45 minutes after the noon. If she was driving at 2 km/min at this moment, how far would be she away from Seoul (along the highway) at 50 minutes after the noon? It is easy. During 5 minutes, she would have driven $10(= 2 \times 5)$ km. So, she should be 160 km away from Seoul. This may be a good answer, but there is a hidden assumption: she was driving at a constant speed. If her speed noticeably changed during the 5 minutes, she would not have driven 10 km during the 5 minutes. Nevertheless, assuming that her speed didn't change much during the 5 minutes, our answer is good, *approximately.*

Let's translate our answer into a mathematical language. If her position at time t is given by $x(t)$. We have

$$x(45) = 150, \qquad x'(45) = 2. \tag{1}$$

Then, we obtained

$$x(50) \approx x(45) + (50 - 45) \times x'(45). \tag{2}$$

More abstractly, we can write

$$x(t') \approx x(t) + (t' - t) \times x'(t). \tag{3}$$

More generally, changing the notation slightly, for any function $f(x)$, we can write

$$f(x) \approx f(x_0) + (x - x_0) \times f'(x_0). \tag{4}$$

This is called a "linear approximation" as we approximated a function (i.e., left-hand side) by a linear function (i.e., right-hand side). This approximation is valid if $f'(x)$ does not change dramatically between x_0 and x considered above. In fact, if $f'(x)$ is a continuous function, the above approximation must hold very well, as long as x is sufficiently close to x_0, because the change of $f'(x)$ can be made sufficiently small by choosing the value x sufficiently close to x_0.

To understand what I mean, see Fig. 1. The right-hand side of (4) is a straight line if we draw it as a graph, and any line, such as $f(x)$ can be approximated as a straight line, which has a constant slope, if you look at closely. The slope, given by $f'(x)$ does not change much in a very small range.

Also, it may seem redundant, but let's derive (4) from the definition of derivative. We have

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{5}$$

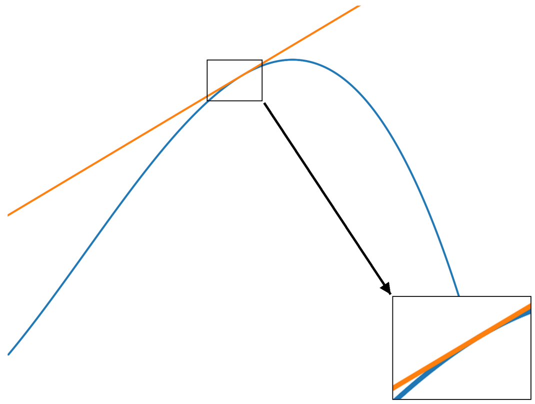

Figure 1: If you look closely, the orange line (the linear approximation) almost coincides with the blue line (the original function).

For x sufficiently close to x_0, we have

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0}. \tag{6}$$

which leads to (4).

Before closing this article, let's ask a question. What should we do, if $f'(x)$ changes strongly? Should we give up the approximation? In our example, that would be the case when Jordanna's speed changes dramatically. In such a case, we can approximate her *speed* by a linear approximation. For this, we would need to know the rate of her speed change (i.e., acceleration) 45 minutes after the noon.[1] Then, we can use this acceleration to approximate the position better, as long as *acceleration* does not change drastically. We already discussed how to obtain the position when we have a *constant* acceleration. In this refined approximation, the position can be expressed as a quadratic function of time. In other words, $f(x)$ can be expressed in terms of a quadratic function of $(x - x_0)$ with coefficients given by $f'(x_0)$ and $f''(x_0)$. Then, this approximation will be better than the linear approximation. What if acceleration changes dramatically, too? Then, we can consider $f'''(x_0)$ and approximate $f(x)$ by a cubic function and so on. Each step will render the approximation better. We will talk more about it in our article "Taylor series."

Problem 1. In our example, where would be Jordanna's approximate position 43 minutes after the noon?

Summary

- A linear approximation is given by $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.

[1]This is not the exact definition of acceleration, because acceleration is the rate of change of velocity, which includes direction. However, for simplicity, we will refer to the rate of change of speed as acceleration, assuming that the highway is straight.