Linear independence, linear dependence and basis

In our earlier article "System of linear equations, part II: three or more unknowns," we have given loose definitions for linear independence and linear dependence. In this article, we will give you their precise mathematical definitions.

A set of n vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ is called "linearly independent," if and only if the only solution to the following equation is $c_1 = c_2 = \cdots = c_n = 0$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0 \tag{1}$$

On the other hand, a set of n vectors is called "linearly dependent," if and only if it is not linearly dependent. In other words, if there exists a solution to the above equation such that not all cs are zero (i.e. at least one of cs is non-zero), then it is linearly dependent. Notice that in such a case, we can express at least one of the set of vectors in linear combination of other vectors. For example, let's say $c_i \neq 0$ for some i. There exists such an i as at least one of cs is non-zero. Then, we can write,

$$v_i = -\frac{c_1}{c_i}\vec{v}_1 - \frac{c_2}{c_i}\vec{v}_2 - \cdots - \frac{c_{i-1}}{c_i}\vec{v}_{i-1} - \frac{c_{i+1}}{c_i}\vec{v}_{i+1} - \cdots - \frac{c_n}{c_i}\vec{v}_n \tag{2}$$

This completes the proof.

Problem 1. Determine whether the following sets of vectors are linearly dependent or independent. Hint: try to solve (1)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{5}$$

There is an alternative definition to linear independence. Let's say a set of vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent, and $\vec{u}$ can be expressed as a linear combination of these vectors as follows:

$$\vec{u} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \tag{6}$$

Then, xs are unique, given $\vec{v}$. Let's prove this. Assume that the following ys also satisfy the above equation. In other words,

$$\vec{u} = y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n \tag{7}$$

Subtracting (7) from (6), we get:

$$0 = (x_1 - y_1)\vec{v}_1 + (x_2 - y_2)\vec{v}_2 + \cdots + (x_n - y_n)\vec{v}_n \tag{8}$$

Since vs are linearly independent,

$$0 = x_1 - y_1 = x_2 - y_2 = \cdots = x_n - y_n \tag{9}$$

which implies

$$x_1 = y_1, \quad x_2 = y_2, \quad \cdots \quad , x_n = y_n \tag{10}$$

This is exactly what we wanted.

Now, let's prove that $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_m, \vec{v}_{m+1}, \cdots, \vec{v}_n$, a set of an arbitrary $n > m$ vectors in $\mathbb{R}^m$ (i.e. m-tuple of real numbers) is always linear dependent. To this end, we need to solve (6) and show that there are at least more than one solutions in at least one cases.

Notice that this is a system of m linear equations with n unknowns. As $n > m$ it is under-determination. There are infinitely many solutions as we have seen in the last article.

We now introduce the concept of "span." The span of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ is the set of vectors $\vec{u}$ expressible as the linear combinations $\vec{u} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$. For example, the following $\vec{v}_1$ and $\vec{v}_2$ never span $\vec{u}$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \tag{11}$$

as there is no way to get the third component of $\vec{u}$ which is -2, by adding multiples of $\vec{v}_1$ and $\vec{v}_2$ as their third components are identically 0.

Now, let's prove that n vectors, where $n < m$, never span $\mathbb{R}^m$. It is enough to show that there exists at least one $\vec{u}$ that cannot be expressed as (6). Notice that this is a system of m equations with n unknowns. If we make it into an echelon form, it will be necessary something like

$$\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & \tilde{u}_1 \\ 0 & \mathbf{1} & 0 & \tilde{u}_2 \\ 0 & 0 & \mathbf{1} & \tilde{u}_3 \\ 0 & 0 & 0 & \tilde{u}_4 \end{array}\right] \tag{12}$$

where $\tilde{u}$ is obtained from $\vec{u}$ by row reduction. Now notice here that there is no solution to above equation for $\vec{u}$ that transforms to $\tilde{u}$ of which $\tilde{u}_4$ is non-zero. Therefore, such $\vec{u}$ is not in the span of $\vec{v}$s.

We are now ready to define basis. A set of vectors $(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n)$ in a vector space V is called a basis, if the vectors are linearly independent and any arbitrary vector in the vector space can be expressed as a linear combination of this set. For example, if the arbitrary vector is $\vec{v}$, we have

$$\vec{v} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \tag{13}$$

Of course, the linear independence condition of the basis vectors make the xs in the above equation unique.

It is also easy to see that there are precisely n basis vectors if the vector space is n dimension or $\mathbb{R}^n$. If there were more than n basis vectors, the vectors wouldn't be linearly independent. If there were less than n basis vectors, they do not span $\mathbb{R}^n$; a set of $m < n$ vectors can span at most m dimensional vector space.

Final comment. In $\mathbb{R}^n$ we have basis $\vec{e}$s which you already know, called "standard basis." It is given by,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 1 \end{bmatrix} \tag{14}$$

We will see another example of basis in the next article.

Problem 2. Convince yourself that the standard basis satisfies the definition that basis must satisfy.

Problem 3. Explain why the following set of vectors cannot be a basis for $\mathbb{R}^2$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \tag{15}$$

Problem 4. Explain why a linear operator acting on vectors on a vector space is uniquely determined by how it acts on the basis vector of the vector space.

Problem 5. The "rank" of a matrix A

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \tag{16}$$

is the dimension of vector space spanned by following vectors.

$$\begin{bmatrix} A_{11} \\ A_{21} \\ \cdots \\ A_{m1} \end{bmatrix}, \begin{bmatrix} A_{21} \\ A_{22} \\ \cdots \\ A_{m2} \end{bmatrix}, \cdots, \begin{bmatrix} A_{1n} \\ A_{2n} \\ \cdots \\ A_{mn} \end{bmatrix} \tag{17}$$

Given this, what are the maximum rank and the minimum rank a 1×3 matrix can have? How about a 4×1 matrix? How about a 3×4 matrix? How about a 7×5 matrix?

Problem 6. Can 5 linearly dependent vectors span 5-dimensional space? Explain why or why not.

Summary

- A set of n vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ is called "linearly independent," if and only if the only solution to the following equation is $c_1 = c_2 = \cdots = c_n = 0$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0$$

- A set of an arbitrary $n > m$ vectors in $\mathbb{R}^m$ is always linear dependent.
- n vectors, where $n < m$, never span $\mathbb{R}^m$.
- A set of vectors in a vector space V is called a basis, if any arbitrary vector in the vector space can be uniquely expressed as a linear combination of this set.
- There are precisely n basis vectors if the vector space is n dimension.