Logarithm

Logarithm is a very important concept in mathematics and physics. Logarithm (also called "log") is defined as follows:

$$\text{if } a^x = b, \qquad \text{we have : } \log_a b = x \tag{1}$$

where we call a a "base." Now, let us give you some examples:

$$\log_{10} 1000 = 3 \quad (\text{ as } 10^3 = 1000), \qquad \log_4 2 = \frac{1}{2} \quad (\text{ as } 4^{\frac{1}{2}} = 2) \tag{2}$$

Now here are some properties of logs. First, notice the following:

$$a^b \times a^c = a^{b+c} \tag{3}$$

If we say $a^b = f$, $a^c = g$, we have $a^{b+c} = fg$, which implies:

$$b = \log_a f, \qquad c = \log_a g, \qquad b + c = \log_a(fg) \tag{4}$$

Therefore we conclude:

$$\log_a f + \log_a g = \log_a(fg) \tag{5}$$

Let's derive another identity for logarithms using this formula. If we let $fg = h$, $g = h/f$, we have:

$$\log_a f + \log_a \frac{h}{f} = \log_a h \tag{6}$$

which in turn implies:

$$\log_a h - \log_a f = \log_a \frac{h}{f} \tag{7}$$

Problem 1. Show the following:

$$\log_a(p^2) = 2\log_a p, \qquad \log_a(p^3) = 3\log_a p$$

Do you see the pattern? In general, $\log_a p^n = n\log_a p$. Let's rigorously prove this. Notice the following

$$(a^m)^n = a^{mn} \tag{8}$$

If we let $a^m = p$, we have:

$$\log_a p = m, \qquad \log_a(p^n) = mn \tag{9}$$

which implies:

$$\log_a(p^n) = n\log_a p \tag{10}$$

In engineering, a logarithm with base 10, called a "common logarithm," is useful. Many even go on to omit the marker for the base in such a case. For example, $\log 10000 = 4$, $\log 0.01 = -2$.

In math and physics, it turns out that a logarithm with a base a number called "e" is useful. In mathematics, e is as important number as π and given by $2.718\cdots$. We call a logarithm with base e a "natural log" and denote it by ln, after the French "logarithme naturel." For example, $\ln e = \log_e e = 1$. Some mathematicians and physicists use the symbol log in place of ln. For example $\log e = 1$. Therefore it can sometimes be confusing whether log means $\log_{10}$ or ln.

In any case, why the number e is special and often used as the base for logarithms is explained in my article "Exponential functions and natural logs."

Some historical remarks. John Napier, a Scottish mathematician and astronomer, who published "Description of the Marvelous Canon of Logarithms" (in Latin) in 1614 is regarded as the first discoverer of the logarithms. This discovery was very useful at the time, when scientific calculators were not available, because logarithms can change multiplication into addition, which is much easier than multiplication. It was especially useful for astronomy, because complicated multiplications were common. Let me explain what I mean by changing multiplication into addition. Suppose you want to calculate 3.456789×1.234567. Then, you can look up a table that gives the logarithms of numbers. By looking at a table that lists number and its logarithms, you can find

$$\log 3.456789 \approx 0.538673, \qquad \log 1.234567 \approx 0.091515 \tag{11}$$

Then, from the property of logarithms, we know

$$\begin{aligned} \log(3.456789 \times 1.234567) &= \log 3.456789 + \log 1.234567 \\ &\approx 0.538673 + 0.091515 = 0.630188 \end{aligned} \tag{12}$$

Now, you look at the table again to find which number has 0.630188 as its logarithms. It is 4.267642. The actual value for the multiplication $3.456789 \times 1.234567 \approx 4.267638$, which is very close to the obtained value, using Napier's method.

Actually, Napier's table had 10 million entries, which took him 20 years of calculation. In fact, he used 0.9999999 as his base, instead of 10. Notice that the choose of base doesn't affect the result of multiplication as (5), which we used in (12), is satisfied for any base a.

Anyhow, when I learned that Napier had used logarithms to do multiplication, I naturally assumed that he used 10 as base, as it is the most common base besides e, which had not yet been discovered, and wondered how he would have been able to find the logarithms with base 10 by hand, as the result would be almost always non-integers. Now, I know that he used 0.9999999 as base, which often results in a big enough number as an answer, which enables us to approximate the answer as the closest integer, without much big error. For example,

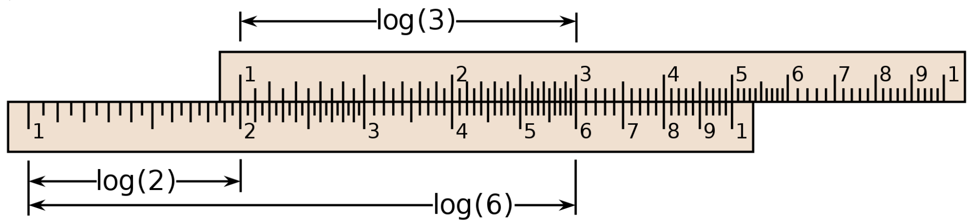
$$\log_{0.9999999} 2.55 = -9360933.129\cdots \tag{13}$$

which is not that much different from the nearest integer −9360933. Indeed, we have

$$0.9999999^{-9360933} = 2.549999968\cdots \tag{14}$$

which is very close to the oringal value 2.55. We already talked about Napier's method in more details in our earlier essay "How to turn a complicated multiplication into a simple addition."

Napier's method for multiplication is often used (in a slightly different form, but essentially using the same principle) as late as 1970s, when my uncle, an engineering student then, used "slide rules." A slide rule is a type of ruler, which enables you to multiply numbers using Napier's method. Below you see an illustration that shows how a slide rule works. Figure out yourself how a slide rule works from the illustration.



Problem 2.

$$\log_4 1 = ?, \qquad \log_2 16 = ?, \qquad \log_{10} 0.1 = ?, \qquad \log_3 \frac{1}{9} = ?, \qquad \log_{100} 10 = ?$$

Problem 3. (Hint[1])

$$\log_8 4 = ?, \qquad \log_3 8 + \log_3 \frac{9}{8} = ?, \qquad \log_{1/2} 4 = ?, \qquad e^{2\ln 3} = ?, \qquad e^{3\ln 2} = ?$$

Problem 4. Prove the following. (Hint[2])

$$\log_a b = \log_a c \cdot \log_c b \tag{15}$$

Notice that this implies $\ln b = \ln 10 \cdot \log_{10} b \approx 2.303 \log_{10} b$

Problem 5. Prove the following.(Hint[3]) Notice that this is useful when calculating the values for $\log_a b$ where a is neither 10 or e, since most calculators provide log values only for these two numbers.

$$\log_a b = \frac{\log_{10} b}{\log_{10} a} = \frac{\ln b}{\ln a} \tag{16}$$

Problem 6. Let's say that a population of a certain country grows by 7 percent every year. Find out how long it takes for it to double, using the logarithm button in a scientific calculator such as the one provided by Microsoft Windows. If you do this correctly, you should get about 10 years. Repeat the calculation for the annual growth rates of 5 percent, 3 percent, 2 percent and 1 percent. If you do this correctly, you should get about 14 years,

[1] For the first one, try to use (8) with n the answer we want, $a = 2$, $a^m = 8$ and $a^{mn} = 4$.

[2] Let $a^x = b$, $a^y = c$ and $c^z = b$, and try to derive $x = yz$

[3] This is the solution to $a^x = b$. Obtain the solution by taking $\log_{10}$ or ln on both-hand sides. You will need to use (10).

23 years, 35 years, 70 years respectively. Notice that the annual growth rate multiplied by the years that takes the population to double is roughly given by 70. For example, $7 \times 10 \approx 5 \times 14 \approx 3 \times 23 \approx 2 \times 35 \approx 1 \times 70 \approx 70$. This is known as the "law of seventy," which I learned from an economics textbook. Using this law, one can easily calculate how long a quantity with a fixed annual growth rate takes to double. For example, if an annual growth rate is 4 percent, it would take about $18(\approx 70 \div 4)$ years to double. If an annual growth rate is 0.5 percent, it would take about $140(= 70 \div 0.5)$ years to double. However, the law of seventy does not hold for big annual growth rates. For example, if the annual growth rate is 100 percent, it takes exactly one year to double, not $0.7(= 70 \div 100)$ years to double. Anyhow, in our later article "Exponential functions and natural logs," you will be invited to prove the law of seventy. You will need to use the natural log.

Problem 6. Figure out how to calculate the square root of a number using Napier's method and table. (Hint[4])

(The slide rule illustration is from `https://commons.wikimedia.org/wiki/File:Slide_rule_example2_with_labels.svg`)

Summary

- If $a^b = c$, then $\log_a c = b$. a is called a "base."
- $\log_a(fg) = \log_a f + \log_a g$
- $\log_a(f/g) = \log_a f - \log_a g$
- $\log_a(f^n) = n \log_a f$
- A logarithm with base 10 is called a "common logarithm" and often denoted as log, without the marker for the base.
- A logarithm with base e is called a "natural log" and often denoted as ln.

[4] First, express $\log \sqrt{x}$ in terms of $\log x$.