Mass-energy equivalence

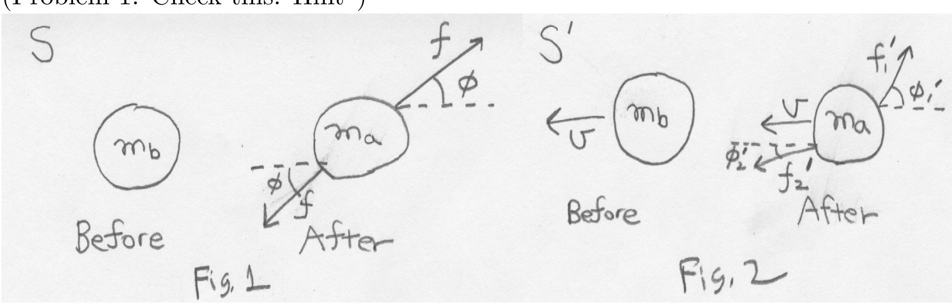
In an earlier article "Relativistic momentum and energy," we have introduced the famous equation "$E = mc^2$." However, the way we introduced may not be convincing enough to understand that the mass is indeed equivalent to energy. From this consideration, we show that it indeed is by closely following Einstein's original paper in 1905. See Fig. 1. In the reference frame S, an object with mass m_b is at rest. However, after the emission of two photons with frequency f and directions making an angle ϕ with the x-axis, it loses energy $2hf$ and now has a smaller mass m_a as mass is equivalent to energy. We will find the difference between m_b and m_a in terms of the lost energy. In any case, if we write the energy of the object before the emission as E_b and the one after the emission E_a, the conservation of energy says:

$$E_b = E_a + 2hf \tag{1}$$

Now, see Fig. 2 for the reference frame S' which moves in positive x-direction with a speed v with respect to S. The object still has the mass m_b before the emission and still has the mass m_a after the emission, if v is small enough to ignore the relativistic effect of mass increase. However, the change in the frequencies of emitted photons cannot be ignored, for our purpose. Using the Lorentz transformation for the wave number and angular frequency derived in our last article, we have:

$$f_1' = \frac{f(1 - v\cos\phi/c)}{\sqrt{1 - v^2/c^2}}, \qquad f_2' = \frac{f(1 + v\cos\phi/c)}{\sqrt{1 - v^2/c^2}} \tag{2}$$

(Problem 1. Check this. Hint[1])



Therefore, if the energy of the object before the emission is E_b' and the one after the

[1]Find a relation between f and ω. Then, express $k \equiv \sqrt{k_x^2 + k_y^2 + k_z^2}$ in terms of ω and c. Express also k_x in terms of ϕ and k.

emission is E'_a in the reference frame S', the conservation of energy says:

$$E'_b = E'_a + hf'_1 + hf'_2 = E'_a + \frac{2hf}{\sqrt{1 - v^2/c^2}} \approx E'_a + 2hf\left(1 + \frac{v^2}{2c^2}\right) \tag{3}$$

Now, notice that E'_b is bigger than E_b by the kinetic energy of the object. As the mass of the object is given by m_b, we have

$$E'_b = E_b + \frac{1}{2}m_b v^2 \tag{4}$$

Similarly, E'_a is bigger than E_a by the kinetic energy of the object, as in the reference frame S the object after the emission is still at rest, since the total momentum of the photons emitted is zero as they are emitted in opposite directions, with the same frequency. Therefore, we have:

$$E'_a = E_a + \frac{1}{2}m_a v^2 \tag{5}$$

If we combine (1), (4) and (5), we get:

$$E'_b = E'_a + \frac{1}{2}(m_b - m_a)v^2 + 2hf \tag{6}$$

Comparing this with (3), we conclude:

$$m_b - m_a = \frac{2hf}{c^2} \tag{7}$$

Now, see Fig. 1. the object lost $2hf$ of energy, thus the mass decreased by $2hf/c^2$. In other words, the object's mass decreased by $1/c^2$ of energy lost. In other words,

$$\Delta m = -\Delta E/c^2 \tag{8}$$

This is exactly $E = mc^2$. Mass and energy are indeed equivalent. In our case, the two photons, whose rest mass are zero, carry the total relativistic mass $2hf/c^2$ as their energy is $2hf$. By the conservation of mass, we have $m_b = m_a + 2hf/c^2$. There is another way of seeing this. From $E = mc^2$, the concerned object has energy $E_b = m_b c^2$ before the emission and $E_a = m_a c^2$ after the emission. By the conservation of energy, we have $E_b = E_a + 2hf$, which implies $m_b = m_a + 2hf/c^2$.

Summary

- Mass and energy are equivalent. $E = mc^2$.