The mathematical beauty of physics: simplicity, consistency, and unity

A British physicist Paul Dirac says of mathematical beauty as following:

“It is quite clear that beauty does depend on one's culture and upbringing for certain kinds of beauty, pictures, literature, poetry and so on...But mathematical beauty is of a rather different kind. I should say perhaps it is of a completely different kind and transcends these personal factors. It is the same in all countries and at all periods of time.”

The law of physics posseses the mathematical beauty Dirac talked about. So, what is the mathematical beauty of physics? I propose that the law of physics is mathematically beautiful because it has three qualities: simplicity, consistency, and unity.

First, simplicity. As examples, I will give you Coulomb’s law, Newton’s law of universal gravitation and Maxwell’s equations. Coulomb's law states that the Coulomb force (i.e., an electric force) is inversely proportional to the square of the distance between the two charges. For example, if the distance between two charges is doubled, the force is quartered. If the distance is tripled, the force becomes one ninth its previous value. Mathematically, we can express this as

$$\text{Coulomb force} \propto \frac{1}{r^2}$$

where r denotes the distance between the two charges. In other words, we say Coulomb's law obeys “inverse-square” property.

If we repeat now Coulomb's experiments in the 18th century, will we exactly get that the Coulomb force is inversely proportional to the square of the distance? To answer this question properly, you need to understand that there is an error whenever you measure something. For example, if the smallest scale in your ruler is 1 millimeter, at best you can measure something with that ruler within an error of 0.5 millimeter. For example, if something is 156.3248 millimeters long, at best you will measure it to be 156 millimeters i.e., an error of 0.3248 millimeters. As there are errors in the measurement, there should be an error in Coulomb's law. For example, in 1971, it was found that

$$\text{Coulomb force} \propto \frac{1}{r^{2+q}}$$

where $q = (2.7 \pm 3.1) \times 10^{-16}$. If you do not know this kind of notation, it means

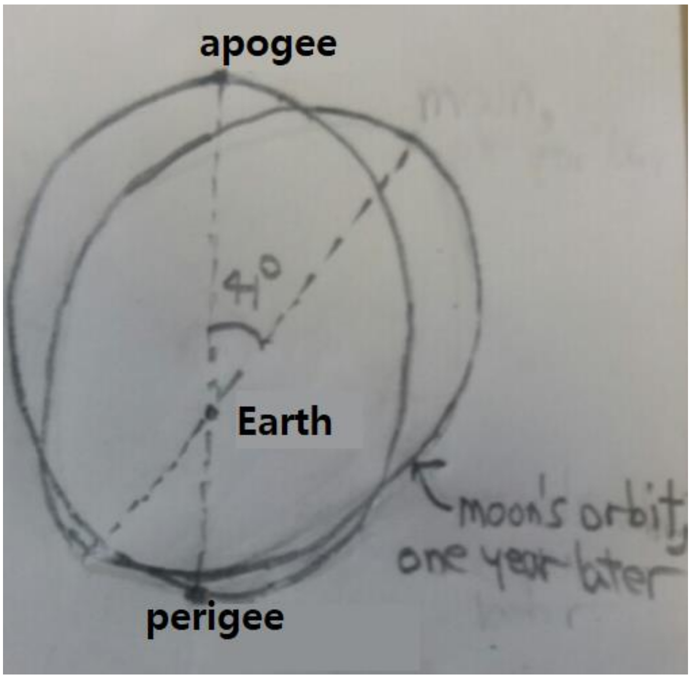
$$\frac{1}{r^{2.00000000000000058}} < \text{Coulomb force} < \frac{1}{r^{1.99999999999999994}}$$

In other words, the Coulomb force obeys inverse-square property to a very high accuracy. Actually, there is another force that obeys inverse-square property to a very high accuracy: Newton’s universal gravitation.

Newton's law of universal gravitation says that the gravitational force between two objects is proportional to the product of the masses of each object and its dependence on the distance satisfies inverse square property. This gravitational force applies to any objects with mass. By proposing this law, Newton succeeded in explaining diverse and seemingly-unrelated phenomena such as an apple falling to the ground, the planets orbiting around the sun, the moon orbiting around the earth, and the ebb and flow of tides caused by the moon and so on.

However, consider the case there are three objects. For example, the Sun, the earth and the moon. The moon is attracted by the earth due to gravitation, but it is also attracted by the Sun. So, the total force acting on the moon is the sum of the gravitational force from the Sun and the gravitational force from the earth. (Of coure, other planets attract the moon as well, but their forces are negligible in considering our problem.) Similarly, the total force acting on the Earth is the sum of the gravitational force from the Sun and the gravitational force from the moon. So, analyzing the motions of the earth and the moon is quite complicated; their motion is intricately related. The theory explaining the motion of the moon is called "lunar theory."

At the crudest approximation, the orbit of the moon relative to the earth is an ellipse (i.e., a squeezed circle). We will show this in later articles. However, it is not an exact ellipse; the direction of the line connecting perigee (the moon's position when it's closest to the earth) and apogee (the moon's position when it's farthest from the earth) rotates at the rate of about 41 degrees per a year. This is known as the rotation of Moon's apsides. See the figure. (The figure is not in scale. The actual distance to the moon doesn't change as much as in the figure.)



(If moon's orbit were an exact ellipse, it means that the dotted line doesn't rotate.)

Newton tried to calculate these 41 degrees himself, but only got half of its value. After more mathematical technics had been developed after Newton's death, some of the best mathematicians such as Leonhard Euler, Alexis Clairaut, and Jean le Rond d'Alembert attacked this problem again in 1740s but they also got half of the correct value as Newton did. Therefore, Clairaut considered the possibility that gravitational force didn't follow inverse square law property. He considered adding a k/r^3 term as follows:

$$\text{Gravitational force} \propto \frac{1}{r^2} + \frac{k}{r^3}$$

Of course, k should be very small so that the gravitational force doesn't deviate from the inverse square law much. Otherwise, the phenomena that Newton had explained using his inverse square law wouldn't be still explained within observational errors. On the other hand, k should be big enough so that it can explain the 41 degrees.

The news of Clairaut's work reached other mathematicians. For example, in 1748, a review of Clairaut's work was published in Journal de Trévoux. However, retracting his earlier position, Clairaut announced in 1749 that he succeeded in explaining the Moon's apsides problem from Newton's inverse square law; he found that some of the terms he had neglected in his previous calcuation was actually important. But, he kept the details of his work secret before the official publication.

Upon hearing the news that Clairaut solved the problem, Euler sent him a letter, asking for the method. Clairaut didn't tell him the method, but just replied that he would publish his computation soon. Impatient, Euler suggested the Imperial Academy of Sciences in Saint Petersburg to propose the explanation of Moon's absides problem by Newton's gravitational law for the academic prize of 1750. As soon as his proposal was accepted, Euler wrote to Clairaut encouraing him to participate; as Euler was one of the judges, he would be able to look at their methods before anyone else. Clairaut participated in the contest and Euler soon obtained Clairaut's paper in February 1751. Then he spent the whole of March studying it. Inspired by Clairaut's paper, Euler did his research on his own on the apsidal motion. Clairaut indeed won the prize. In October, Euler wrote to the secretary of the Saint Petersburg Academy whether his own work could be published along with Clairaut's. However, he learned that the Saint Petersburg Academy could publish his work in Berlin. Possibly wanting to be the first one to publish the important result, he wrote the secretary of the Saint Petersburg Academy to give his manuscript back, but the Berlin publication failed, so he had to send the manuscript back to Saint Petersburg again. Finally, it was published in Saint Petersburg in 1753, several months after the publication of Clairaut's paper.

So much diversion. Summarizing, the Coulomb force and Newton's law of universal gravitation obey inverse-square property to a very high accuracy. Is this a coincidence? Or is there a more fundamental reason why it should be so?

Actually, it turns out that Coulomb's law can be re-expressed by an equation now called "Gauss's law." An interesting fact is that Gauss's law can only express forces that obey the inverse square property. If the Coulomb force were proportional to 1/r^2.0001 or 1/r^3, or 1/r^2+k/r^3 then there would be no simple way to modify Gauss's law to account for it. That the Coulomb force obeys the simple inverse square property, and that it can be re-expressed by Gauss's law show the beauty of physics.

In 1861, Maxwell published "On Physical Lines of Forces" in which he established Maxwell's equations, which were collections of laws that govern everything of the electromagnetic fields. His equations were twelve complicated equations. In 1884, Heaviside reduced them to four simple equations. (One of them was Gauss's law.) In 1908, Minkowski reduced it to two. When I learned in freshman math class homework how to reduce the four Maxwell's equation to the simple two equations, I was mesmerized. "dF=0, d*F=j" This is the simplicity of physical laws. Notice that Maxwell didn't know these simpler and more beautiful versions of his equations, when he first created them. The fact that this simplification has been possible demonstrates that there is "something," a mathematical reality.

In physics, you often encounter such simplicities. It is just amazing how after a long night of calculations every ugly term cancels out and a complicated string of symbols boils down to an equation a single line long. This simplicity appears like a miracle from God. Maybe God truly is behind it. I think this must be what Albert Einstein meant in saying, "When the solution is simple, God is answering."

Second, The consistency of physics—the ability to arrive at the same conclusion using different methods—seems to me to be another one of God's miracles. For example, Heisenberg's quantum mechanics and Schrödinger's quantum mechanics are equivalent, even though they may seem very different.[1] The actual calculations involved in each are very different, but amazingly, they give the same results.

Similarly, Feynman's quantum electrodynamics looks quite different from those of Tomonaga and Schwinger, but each set of calculations gives the same answer[2]. Yet another example: Joseph Polchinski writes in the preface of "String Theory" that "...the critical dimension of the bosonic string is calculated in seven different ways in the text and exercises" in his book.

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[1] Schrödinger proved this equivalency after months of work, even though it turned out to be one-line derivation.

[2] Freeman Dyson proved this equivalency.

As Richard Feynman, Nobel Laureate in Physics, says, "Every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics."[3] I find this degree of consistency to be particularly beautiful.

Finally, let me explain about a third beautiful quality of physics: unity. Albert Einstein once said:

“Creating a new theory is not like destroying an old barn and erecting a skyscraper in its place. It is rather like climbing a mountain, gaining new and wider views, discovering unexpected connections between our starting points and its rich environment. But the point from which we started out still exists and can be seen, although it appears smaller and forms a tiny part of our broad view gained by the mastery of the obstacles on our adventurous way up.”

Every new theory in physics must be able to explain new phenomena *in addition to* describing old phenomena that an old theory has already explained adequately. This requirement constrains the development of new theories in a very strong way.

Physics is not a patchwork endeavor in which you can look at the results of new experiments independent from all others, coming up with separate formulas and theories to explain each of them. Albert Einstein once said, ”A theory can be proved by experiment, but no path leads from experiment to the birth of a theory.”[4] You first have to make a theory consistent with the old and verified theories, calculate what kind of experimental results the new theory predicts, and then compare the predicted results with the experimentally attained ones. (In our later essay, “How are theories and laws in physics created?” we will talk more about this.) This is what I mean by the beauty of “unity.”

Carlo Rovelli and Francesca Vidotto wrote in their book:

“Contradiction between empirically successful theories is not a curse: it is a terrific opportunity. Several of the major jumps ahead in physics have been the result of efforts to resolve precisely such contradictions. Newton discovered universal gravitation by combining Galileo’s parabolas with Kepler’s ellipses. Einstein discovered special relativity to solve the “irreconcilable” contradiction between mechanics and electrodynamics. Ten years later, he discovered that spacetime is curved in an effort to reconcile Newtonian gravitation with special relativity. Notice that these and other major steps in science have been achieved without virtually any *new*

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[3] Feynman, Richard. (2001). *The Character of Physical Law* (24th ed., p. 168). Cambridge, Massachusetts: The M.I.T. Press.

[4] The Sunday Times (18 Jul 1976).

empirical data. Copernicus for instance constructed the heliocentric model and was able to compute the distances of the planets from the Sun using only the data in the book of Ptolemy."[5]

Indeed, it's amazing how pure logic can lead to new theories. Regarding general relativity and Yang-Mills theory, which are fundamental to contemporary particle physics, Fundamental Physics Prize Laureate Nima Arkani-hamed said:

"So if you just hand a bunch of theorists the laws of relativity and quantum mechanics they are confident, that if you lock them up in a room, you don't let them look at what the world looks like outside, and just ask what could the world look like, this is what they will come up with."

Although general relativity was discovered in a different fashion, Yang-Mills theory actually was developed without any experimental input.

Johann Wolfgang Goethe said, "Beauty is a manifestation of secret natural laws, which otherwise would have been hidden from us forever." Certainly, if there were no beauty in physics laws, the Yang-Mills theory would have never been discovered, especially since it could not be deduced from experiments.

Let me conclude this essay with two quotes. Freeman Dyson said:

"On being asked what he meant by the beauty of a mathematical theory of physics, Dirac replied that if the questioner was a mathematician then he did not need to be told, but were he not a mathematician then nothing would be able to convince him of it."

Paul Dirac said:

"The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen."[6]

The more you study physics, the more its simplicity, consistency, and unity will convince you of its

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[5] Rovelli, Carlo, and Francesca Vidotto. (2015). *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory* (p. 5). Cambridge University Press.

[6] Paul Adrien Maurice Dirac. *The Relation between Mathematics and Physics*
Lecture delivered on presentation of the JAMES SCOTT prize, February 6, 1939
Published in: Proceedings of the Royal Society (Edinburgh) Vol. 59, 1938-39, Part II pp. 122-129

truth. You will be sure that aliens, had they had a civilization, would have deduced Newton's laws and Einstein's theory of relativity just as we did. I am not sure the same could be said about the facts of biology or psychology. That's why I love physics the most out of all the sciences.