The mathematical definition of vector

Mathematically speaking, a vector is an entity that satisfies the following eight conditions.

1) Vector addition must be associative. (i.e. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$)

2) Vector addition must be commutative. (i.e. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$)

3) Vector addition must have an identity element. (i.e. For any vector $\vec{u}$, there exists an identity element $\vec{e}$ that satisfies $\vec{u} + \vec{e} = \vec{u}$. Namely $\vec{e} = 0$)

4) Vector addition must have inverse elements. (i.e. $\vec{u} + (-\vec{u}) = 0$, where $-\vec{u}$ is the additive inverse of $\vec{u}$.

5) Distributivity must hold for scalar multiplication over vector addition. (i.e. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ for any a)

6) Distributivity must hold for scalar multiplication over field addition. (i.e. $(a + b)\vec{u} = a\vec{u} + b\vec{u}$ for any a and b)

7) Scalar multiplication must be compatible with multiplication in the field of scalars. (i.e. $a(b\vec{u}) = (ab)\vec{u}$ for any a and b)

8) Scalar multiplication must have an identity element. (i.e. $1\vec{u} = \vec{u}$)

It is easy to check that our earlier notion of vector satisfies all the above conditions. For example, if we denote three three-dimensional vectors as follows:

$$\vec{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}, \quad \vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}, \quad \vec{w} = w_x\hat{x} + w_y\hat{y} + w_z\hat{z} \tag{1}$$

Then, the first condition is satisfied since:

$$\begin{aligned} \vec{u} + (\vec{v} + \vec{w}) &= u_x\hat{x} + u_y\hat{y} + u_z\hat{z} + ((v_x + w_x)\hat{x} + (v_y + w_y)\hat{y} + (v_z + w_z)\hat{z}) \\ &= (u_x + v_x + w_x)\hat{x} + (u_y + v_y + w_y)\hat{y} + (u_z + v_z + w_z)\hat{z} \\ (\vec{u} + \vec{v}) + \vec{w} &= ((u_x + v_x)\hat{x} + (u_y + v_y)\hat{y} + (u_z + v_z)\hat{z}) + w_x\hat{x} + w_y\hat{y} + w_z\hat{z} \\ &= (u_x + v_x + w_x)\hat{x} + (u_y + v_y + w_y)\hat{y} + (u_z + v_z + w_z)\hat{z} \end{aligned}$$

The fourth condition is also satisfied since:

$$-\vec{u} = -u_x\hat{x} - u_y\hat{y} - u_z\hat{z} \tag{2}$$

returns zero when added to $\vec{u}$. One can check the other conditions also easily.

If the scalars a, b in 5), 6), and 7) are real numbers, the vector space which the vectors so defined live in is called "real vector space." If they are complex numbers, the vector space is called "complex vector space."

Final remark. In this article, we saw that an array of three numbers can be regarded as a three-dimensional vector. For example, an array of three real numbers can be regarded as three-dimensional real vector, and a set of such vectors is called three-dimensional real vector space, and denoted as $\mathbb{R}^3$. Similarly, an array of n complex numbers can be regarded as an n-dimensional complex vector, which is an element of $\mathbb{C}^n$, the n-dimensional complex vector space. We will later see that infinite-dimensional complex vector space is useful for quantum mechanics.

Summary

- Mathematically, vectors are quantities that satisfie certain reasonable linearity conditions.
- An array of n real numbers can be regarded as a n-dimensional real vector.
- An array of n complex numbers can be regarded as a n-dimensional complex vector.