The Maxwell Lagrangian and its equations of motion

In an earlier article titled "Non-Abelian gauge theory," we introduced the Maxwell Lagrangian. In this article, we will elaborate on that discussion.

Recall that in Maxwell theory (i.e. Abelian gauge theory) the Lagrangian is given by

$$S = -\int d^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1}$$

Using the hodge star operator, the Lagrangian can be re-written in differential form notation as follows:

$$\begin{aligned} S &= -\int \frac{1}{2} F \wedge *F \qquad (2) \\ &= -\int \frac{1}{2} dA \wedge *dA \qquad (3) \end{aligned}$$

That $\frac{1}{2} F_{\mu\nu} F^{\mu\nu}$ is equal to $F \wedge *F$ can be checked by brute force calculation using $*F$ given in Equation 3 of our earlier article "Maxwell's equations in differential forms." A more general proof is given in our article "Vierbein formalism and Palatini action in general relativity."

From $F = dA$, one immediately gets $dF = 0$ since $d(dA) = d^2 A = 0$ as $d^2 = 0$. However, there is no way to get other half of the Maxwell equations, namely $d * F = J$, since J is not in the above Lagrangian. Therefore, in the Lagrangian, we need an extra term that includes J (i.e. J^μ).

Suppose you are God, and you want to write out the extra term. What choice do you have? Since J is a three-form, we need to wedge product it with a one-form, so that we get a four-form which can then be integrated on four-dimensional spacetime. What one-form do we have? The only one-form we have is the electromagnetic potential A. So we write:

$$S = -\int (\frac{1}{2} dA \wedge *(dA) + A \wedge J) \tag{4}$$

Varying the A, we get:

$$\begin{aligned} \delta S &= -\int (\frac{1}{2} d\delta A \wedge *(dA) + \frac{1}{2} dA \wedge *(d\delta A) + \delta A \wedge J) \\ &= -\int (\frac{1}{2} d\delta A \wedge *(dA) + \frac{1}{2} d\delta A \wedge *(dA) + \delta A \wedge J) \\ &= -\int (d\delta A \wedge *(dA) + \delta A \wedge J) \\ &= -\int (-\delta A \wedge d * (dA) + \delta A \wedge J + \text{total derivatives}) \qquad (5) \end{aligned}$$

Noticing the well-known fact that total derivatives don't change the equation of motion, we conclude:

$$d * (dA) = J \tag{6}$$

In other words, $d * F = J$.

Now, to get another insight, let's express the Lagrangian in component forms as follows:

$$S = -\int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu \right) \tag{7}$$

Under gauge transformation $A_\mu \to A_\mu - \partial_\mu \theta$, the action changes by

$$\Delta S = \int d^4x j^\mu \partial_\mu \theta = -\int d^4x (\partial_\mu j^\mu) \theta + \text{total derivatives} \tag{8}$$

Therefore, for the equations of motion not to be changed for an arbitrary θ, we conclude $\partial_\mu j^\mu = 0$, which is the charge conservation. In other words, gauge invariance implies charge conservation!

Summary

- In the presence of source, the Maxwell action becomes

$$S = -\int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu \right)$$

- The gauge invariance $A_\mu \to A_\mu - \partial_\mu \theta$ implies the charge conservation $\partial_\mu j^\mu = 0$.