Möbius strip and Klein bottle

1 Möbius strip

In the last article, we mentioned that a disk is not the only 2-dimensional surface that has exactly one boundary, and promised to introduce in this article another 2-dimensional surface that also has exactly one boundary. That is Möbius strip. See Fig. 1. Notice that it is different from a cylinder. A cylinder is not twisted, but a Möbius strip is twisted. Does this have really one boundary? See another example of Möbius strip in Fig. 2. The yellow line is the boundary of Möbius strip. If you start on any point on the yellow line and move along it in one direction, you will come back to your original point after covering all the points in all the boundary of Möbius strip. Therefore, it indeed has one boundary. The boundary is one piece and all connected. If you take the same steps for one of the boundaries of cylinder, you do also come back to your original point, but the other boundary is not covered and left out. A cylinder has two disconnected components of boundaries. Simply put, a cylinder has two boundaries.

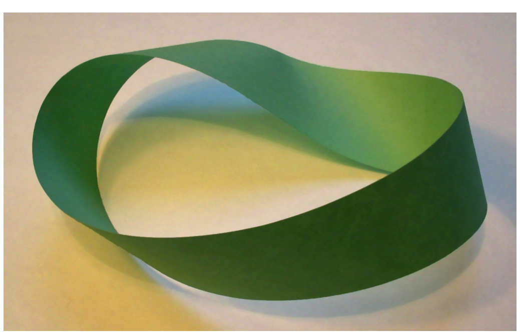

Figure 1: Möbius strip [1]

Figure 2: the yellow line is the boundary of Möbius strip [2]

So, how can we construct a Möbius strip? See Fig. 3. Step 1, you prepare a strip of paper. Step 2, you bend it. Step 3, you twist it once. Step 4, you glue them by a scotch tape in such a manner that A is glued to B and B is glued to A. Notice that if you skip Step 3, and glue A to A and B to B, it would have been a cylinder instead of Möbius strip.

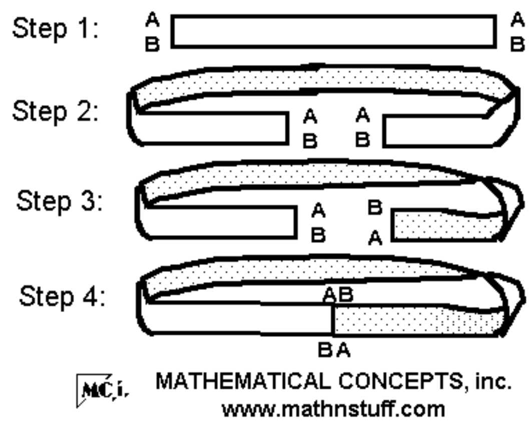

Figure 3: Constructing a Möbius strip [3]

One noticeable feature of Möbius strip is that it has not only one boundary but also only one side. In case of cylinder, you can paint the inside of the cylinder and the ouside of the cyinder in two different colors. See Fig. 3 again. If you skipped Step 3, and constructed a cylinder you would be able to paint the inside as shaded and the outside as white. A cylinder has two sides. However, in case of Möbius strip, as you see in Step 4, what would have been the "inside" of a cylinder and what would have been the "outside" of a cylinder are glued. Therefore, once you begin to shade what you think is inside of the Möbius strip, and continue as you move along the strip, you ended up shading the whole Möbius strip. Mathematicians say Möbius strip is "non-orientable," because it has no inside and outside. On the other hand, mathematicians say that a cylinder is "orientable," because it has the inside and the outside. Before moving onto the next subsection, a quick comment. In an earlier article, we mentioned that there are five string theories. Among these five string theories, Type I string theory admits that the surface string swipes is non-orientable, while all the others do not admit non-orientable string-swiping-surface.

Problem 1. Construct a Möbius strip. Draw a center line in the middle of Möbius strip. Using scissors, cut the Möbius strip along the center line. Surprisingly, you will find that the Möbius strip is not broken into two pieces. Is the resulting strip a Möbius strip? The answer is in Fig. 4.

2 Representations of Möbius strip and cylinder

Is there a simple diagram to represent a Möbius strip? See Fig. 5. If you connect C, D, E on the left side with C, D, E on the right side respectively, you will get a Möbius strip. Thus, Fig. 5 is a good representation of Möbius strip. Similarly, a cylinder can be represented as in Fig. 6.

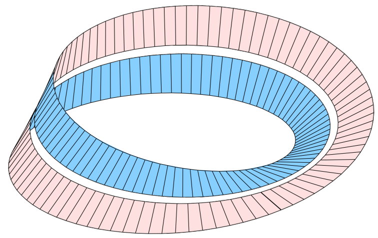

Figure 4: Cutting the middle of a Möbius strip [4]

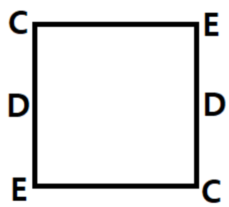

Figure 5: a Möbius strip

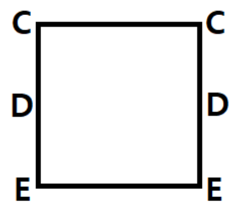

Figure 6: a cylinder

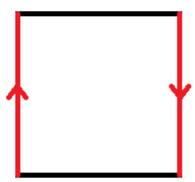

Figure 7: a Möbius strip

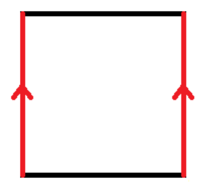

Figure 8: a cylinder

Usually, mathematicians represent Fig. 5 and Fig. 6 as Fig.7 and Fig. 8. I would like to note that what is important in representing Möbius strip and cylinder as in these figures are topology, i.e., connectedness, not how these two-dimensional surfaces are embedded (or, look like) in 3-dimensional space. Therefore, the figures are accurate representations of these two 2-dimensional surfaces.

3 Representations of torus and Klein bottle

Can we represent a torus as in the way we just represented Möbius strip and cylinder? You can make a torus by gluing two boundaries of a cylinder. Thus, if you glue the two black lines in Fig. 8 as two blue lines in Fig. 9, you will get a torus.

However, is this the only way to glue the two boundaries of a cylinder? No. We can reverse one of the arrows and glue them as in Fig. 10. This surface is called "Klein bottle."

Of course, if you try to make Klein bottle with a paper strip, you won't be able to glue the two boundaries of a cylinder in the manner I just mentioned. See Fig. 11 for an attempt

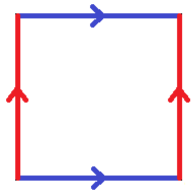

Figure 9: a torus

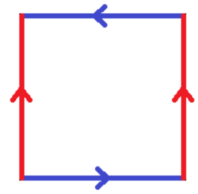

Figure 10: a Klein bottle

to glue them so, and see Fig. 12 to see another 3d visualization of Klein bottle.

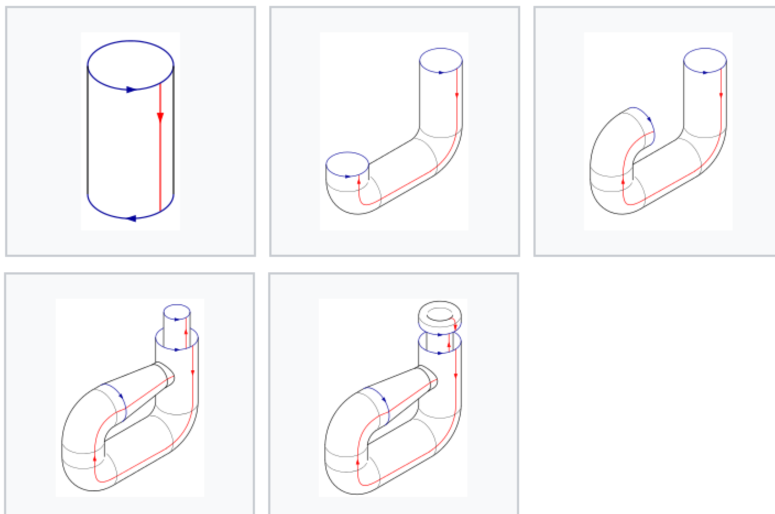

Figure 11: gluing a cylinder to make Klein bottle [5]

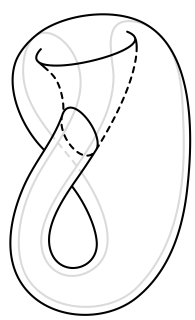

Figure 12: a 3d visualization of Klein bottle [6]

You see that the Klein bottle "crosses itself" in the 3d visualization. In other words, in 3 dimensional space, we cannot draw a Klein bottle. If you see Fig. 10, no surface crosses itself; only two red lines and two blue lines are identified among themselves. However, that doesn't mean that Klein bottle doesn't exist. They exist as much as 3-sphere or 4-manifolds exist.

The only problem we have is that we cannot embed Klein bottle in 3-dimensional space. But, again, as I said in our earlier article "Manifold," two dimensional surfaces exist on their own own without the necessity to be embedded in higher dimensional space. Actually, one can embed Klein bottle in 4-dimensional space by following way. Let's first draw Klein bottle in 3 dimensional space. Then, it crosses itself. If you give different 4th coordinates for the region that Klein bottle is crossing itself in 3 dimensional space, it is no longer crossing itself, because the "crossing" regions are no longer located at the same location because their 4th coordinates are different.

When I was little, I read about Klein bottle in a math cartoon book. I never understood what it was. Only after I saw Fig. 10 much later, could I understand what Klein bottle is.

Like Möbius strip, Klein bottle is non-orientable. It has no inside and outside. On the other hand, torus is orientable.

Can all two-dimensional surfaces be embedded in 4-dimensional space? When I was a freshman at Harvard, I tried to take a course title "Differential topology," even though I didn't have the required prerequisites. It was listed as a course that would be helpful to understand string theory. I ended up dropping it, but I learned that n-dimensional manifold can be always embedded in $2n$-dimensional space. So, the answer is yes.

Summary

- A Möbius strip has only one boundary and one side. Whereas a cylinder has two boundaries and two sides.
- Möbius strip, cylinder, torus and Klein bottle can be represented by a square with arrows on it.
- A two-dimensional surface called "Klein bottle" cannot be embedded in 3 dimensional space, but can be in 4 dimensional space.

References

[1] https://commons.wikimedia.org/wiki/File:M%C3%B6bius_strip.jpg

[2] https://commons.wikimedia.org/wiki/File:MobiusJoshDif.jpg

[3] http://www.mathnstuff.com/papers/tetra/moebius.htm

[4] https://commons.wikimedia.org/wiki/File:Moebiusband-1s.svg

[5] https://en.wikipedia.org/wiki/Klein_bottle

[6] https://commons.wikimedia.org/wiki/File:Surface_of_Klein_bottle_with_traced_line.svg