Moment of inertia revisited

Suppose an object is rotating around an axis with angular velocity ω. The question we will answer in this article is: "How can we express the kinetic energy of the rotating object?"

To this end, think along this way. The kinetic energy of the rotating object is the sum of the kinetic energy of each fraction that comprises the whole rotating object. Then, to find the kinetic energy of each fraction, we need to find their mass and their speed.

Now, notice that the speed is given by $r\omega$ if r is the distance to the axis. We have already seen this in our earlier article, and it indeed makes sense as the farther each fraction is away from the axis the faster its speed is. If the mass of the fraction concerned is given by m, its kinetic energy is given by $\frac{1}{2}m(r\omega)^2$. Now, if we sum the kinetic energy of each fraction, we get the total kinetic energy as follows:

$$K = \sum_i \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2 \tag{1}$$

If we define the term in the parenthesis by I, the kinetic energy is given by

$$K = \frac{1}{2} I \omega^2 \tag{2}$$

where

$$I = \sum_i m_i r_i^2 \tag{3}$$

If we turn the sum into an integral by making m_i infinitesimal, we have:

$$I = \int dm r^2 \tag{4}$$

This is the definition of moment of inertia. Now, let's calculate the moment of inertia for

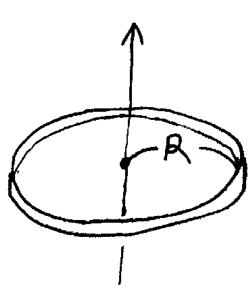

Figure 1: a ring with radius R

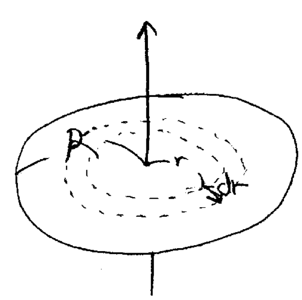

Figure 2: a disc with radius R

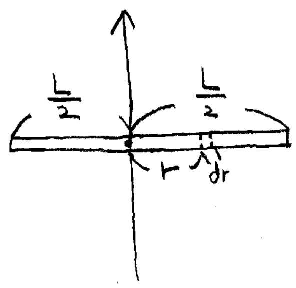

Figure 3: a rod with length L

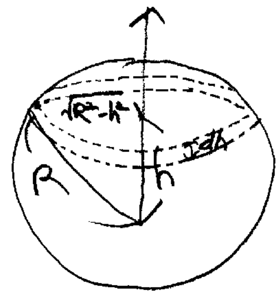

Figure 4: a ball with radius R

some examples. See Fig. 1. We have a ring with mass M and radius R. As all the fractions of the ring is distance R away from the axis, we have:

$$I = \int dm\, r^2 = \int dm R^2 = R^2 \int dm = MR^2 \tag{5}$$

See Fig. 2. for our second example. We have a uniform disc with radius R and mass M. Being uniform means that the mass is proportional to the area. If we call the surface mass density σ then $\sigma = M/(\pi R^2)$. Now notice that the area inside the dotted region is $2\pi r dr$ since the circumference is $2\pi r$ and the width is dr. Therefore, the mass of this region is $2\pi\sigma r dr$. So, we have:

$$I = \int dm\, r^2 = \int_0^R 2\pi\sigma r dr r^2 = 2\pi\sigma \frac{R^4}{4} = \frac{1}{2} MR^2 \tag{6}$$

See Fig. 3. for our third example. We have a uniform rod with length L and mass M. Therefore, the linear mass density is given by $\lambda = M/L$. Then, the mass inside the dotted region is given by λdr. So, we have:

$$I = \int dm\, r^2 = \int_{-L/2}^{L/2} \lambda dr r^2 = \lambda \frac{L^3}{12} = \frac{1}{12} MR^2 \tag{7}$$

See Fig. 4. for our fourth example. We have a uniform ball with radius R and mass M. As the volume of ball with radius R is $\frac{4}{3}\pi R^3$, the mass density is given by $\rho = \frac{3}{4}\frac{M}{\pi R^3}$. Now, we can regard the ball as a sum of discs. Each disc has radius $\sqrt{R^2 - h^2}$ from Pythagorean theorem and width dh, where h varies from $-R$ to R. The mass of each disc is given by $dM = \rho\pi(\sqrt{R^2 - h^2})^2 dh$, and its moment of inertia $dI = \frac{1}{2} dM(\sqrt{R^2 - h^2})^2$ from (6). Now, all we are left to do is summing the moment of inertia of each disc. We have:

$$I = \int dI = \frac{1}{2} \int dM(R^2 - h^2) = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - h^2)^2 dh = \frac{2}{5} MR^2 \tag{8}$$

We see that this example is hardest among the ones that I have presented. Don't worry that this is so hard. It was hard for me as well when I first learned it. Now, it's easy for me.

Problem 1. See Fig. 5. We have a uniform ring with mass M and inner radius R_1 and outer radius R_2. What is the moment of inertia? After obtaining the answer check that your

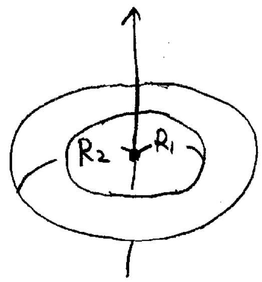

Figure 5: a ring with inner radius R_1 and outer radius R_2

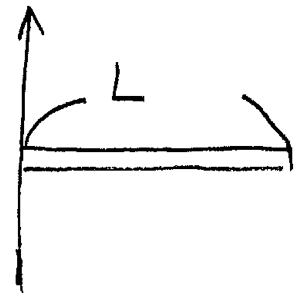

Figure 6: a rod with length L

Figure 7: a cylinder with radius R and height h

answer is correct by comparing your answer with our first example when $R_1 = R_2 = R$ and our second example when $R_1 = 0, R_2 = R$.

Problem 2. See Fig. 6. We have a uniform rod with mass M and length L. Find the moment of inertia.

Problem 3. See Fig. 7. We have a uniform cylinder with mass M, radius R and height h. Find the moment of inertia.

Summary

- The moment of inertia is given by

$$I = \int dm\, r^2$$