Motivating integration

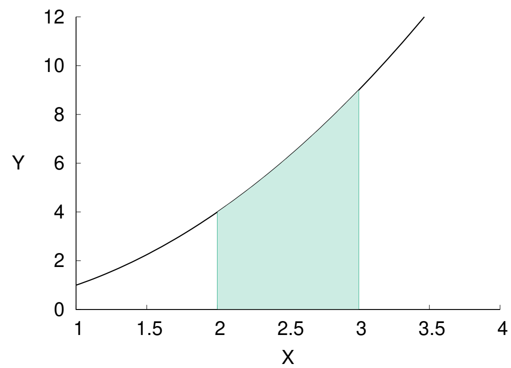

Figure 1: Representation of the function $f(x) = x^2$. The area of the region we want to calculate is highlighted in cyan.

See Fig. 1. You see a graph $y = x^2$, and you want to find the area bounded by $x = 2$, $x = 3$, $y = 0$, and $y = x^2$. How can you find it? If you think about the lesson you learned in the last article, you can do it. See Fig. 2. We divided the range $2 \leq x \leq 3$ into 10 equal steps. As before, we can approximate the shaded area in Fig. 1 by the total area of rectangles in Fig. 2. Let's call it A_{10-}, where 10 denotes the number of ranges divided. We have

$$\begin{aligned} A_{10-} &= (2^2 + 2.1^2 + 2.2^2 + \cdots + 2.7^2 + 2.8^2 + 2.9^2) \times 0.1 \quad (1) \\ &= \frac{20^2 + 21^2 + 22^2 + \cdots + 27^2 + 28^2 + 29^2}{10^2} \times \frac{1}{10} \quad (2) \\ &= \frac{1}{10^3} \sum_{k=20}^{29} k^2 = \frac{1}{10^3} \sum_{k=2\times 10}^{3\times 10-1} k^2 \quad (3) \end{aligned}$$

More generally, if we divide $2 \leq x \leq 3$ into n equal range, we get

$$A_{n-} = \frac{1}{n^3} \sum_{k=2n}^{3n-1} k^2 \quad (4)$$

Problem 1. By using $\sum_{k=1}^{s} k^2 = s(s+1)(2s+1)/6$, show that

$$\lim_{n\to\infty} A_{n-} = \frac{19}{3} \quad (5)$$

This is the answer for the shaded area in Fig. 1. However, we can solve this problem slightly differently.

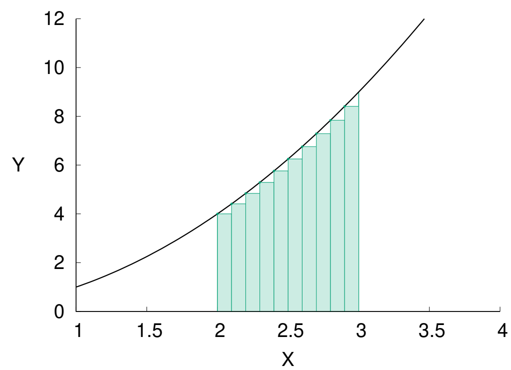

Figure 2: Idem as Fig. 1, together with the rectangles used to generate the sum. The rectangles fall below the curve, only touching on the upper left vertex.

Problem 2. See Fig. 3. Let's call the total area of rectangles by A_{10+}, where 10 denotes the number of ranges divided. Then, we have

$$A_{10+} = (2.1^2 + 2.2^2 + \cdots + 2.7^2 + 2.8^2 + 2.9^2 + 3.0^2) \times 0.1 \tag{6}$$

By taking the similar steps as in (2), (3) and (4), show that

$$\lim_{n\to\infty} A_{n+} = \frac{19}{3} \tag{7}$$

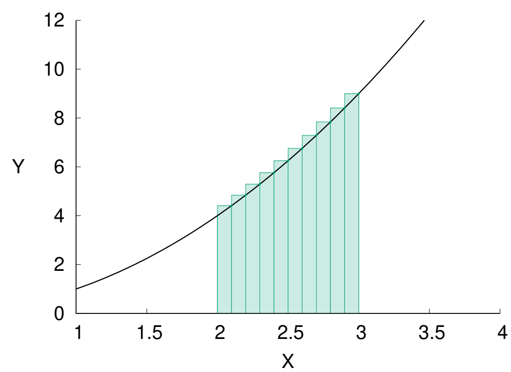

Figure 3: Idem as Fig. 2, but now the rectangles touch the curve with the upper right vertex.

Thus, we see that it doesn't really matter which of the two approximation methods we take.

In this article, we obtained the area under $y = x^2$ for the range $2 \le x \le 3$. In physics, mathematics and engineering, we often encounter similar problems: finding the area under a graph (i.e., a function) for a given range. Precisely speaking, problems that can be translated into finding the area under a function for a given range. Such problems are called "integration." As an example, remember that we mentioned in the last article that the volume of a cone can be also obtained easily using integration without using summation. We will concretely see this later.

In this article, we "integrated" a function as simple as $y = x^2$, using a formula for the sum of squares. Thus, we could calculate the area easily, because we know how to calculate the sum of squares. However, how would we find the area under more complicatd functions such as $y = \tan x$ or $y = 1/x$? There are no simple expressions for the sum of $\tan k$ or $1/k$. However, it turns out that you can calculate the areas under such functions, *without* knowing the expressions for the sum of such functions. Actually, it turns out that finding such areas is *easier* than finding the expressions for the sum, which is impossible in many cases. We will see how we can integrate functions in the next article.

Summary

- Finding the area under a function is called "integration."