Pythagoras and the musical scale

If you hear Do and Sol at the same time, the sound is pleasing, but if you hear Do and Re at the same time, the sound is not pleasing. Why is it so? Pythagoras found out the reason. Let me ask another question. If you know something about the musical scale, you will know that there is a semitone between Mi and Fa, and between Si and Do. Except for these two cases, there is a full tone difference between adjacent notes. Why is it so? We will answer these two questions in this essay.

Harp has a lot of strings stretched in certain lengths. If the lengths are different, the tones are different. In other words, the length determines the tone. The shorter the length, the higher the tone. Pythagoras found out that the tone becomes an octave higher if the length is halved. If the length is shortened by the ratio 2/3, the tone becomes perfect fifth higher. For example, Do becomes Sol. What Pythagoras later found out was that, when the length of the strings that correspond to two different notes have a ratio of two very small integers, they sound pleasing. For example, Do and Do, an octave higher, sound pleasing when played together, because the ratio is

$$1 : \frac{1}{2} = 2 : 1 \tag{1}$$

where 2 and 1 are the small integers that I talked about. Similarly, Do and Sol sound pleasing, because

$$1 : \frac{2}{3} = 3 : 2 \tag{2}$$

where 3 and 2 are the small integers.

Summarizing what we just have seen, if we say that the length corresponding to Do is L, then Do, an octave higher, corresponds to $L/2$, and Sol corresponds to $(2/3)L$. Given this, let's find another note that sounds pleasing with Do. Instead of the ratio 2/3, we can consider the ratio 3/2. In other words, $(3/2)L$. However, this tone is lower than the original Do. So, let's consider the one octave higher of this tone because we want to find a scale that is between the original Do and the Do, an octave higher. Thus, we get

$$\frac{3}{2}L \div 2 = \frac{3}{4}L \tag{3}$$

Let's call this tone "Fa." Of course, both $(3/2)L$ and $(3/4)L$ correspond to Fa. The only difference is that $(3/2)L$ is an octave lower.

Now, let's find another note that sounds pleasing with Sol other than Do. In other words, as dividing $(2/3)L$ by 2/3 yields L which is Do that we already have, let's multiply $(2/3)L$ by 2/3. Then, we have $(4/9)L$. In other words,

$$\frac{2}{3}L : \frac{4}{9}L = 3 : 2 \tag{4}$$

which is a simple ratio. However, this note is higher than an octave-higher-Do, because

$$\frac{4}{9}L < \frac{1}{2}L \tag{5}$$

If we confine our search of the tone, between the original Do (i.e., L) and the Do an octave higher (i.e. $L/2$), we should consider the tone that is an octave lower than the one that corresponds to $(4/9)L$. To lower it by an octave, we multiply the length by two, which results in $(8/9)L$. Let's call this note "Re." In other words, $(8/9)L$ corresponds to Re, and $(4/9)L$ corresponds to Re, an octave higher. Re and Sol sound pleasing together as

$$\frac{8}{9}L : \frac{2}{3}L = 4 : 3 \tag{6}$$

where 4 and 3 are small numbers. Now, let's multiply Re, $(8/9)L$ by $2/3$, to find the tone note is pleasing with Re. We get $(16/27)L$. Let's call this tone "La." It is easy to see that the note is in the wanted range, as

$$L > \frac{16}{27}L > \frac{1}{2}L \tag{7}$$

Now, let's repeat the process.

$$\frac{16}{27}L \times \frac{2}{3} = \frac{32}{81}L < \frac{1}{2}L \tag{8}$$

Thus, to lower it by an octave, we have

$$\frac{32}{81}L \times 2 = \frac{64}{81}L \tag{9}$$

Let's call this note "Mi." If we repeat once more, we get

$$\frac{64}{81}L \times 23 = \frac{128}{243}L \tag{10}$$

Let's call this "Si."

Now, let's summarize what we have just found. Table. 1 is called the Pythagorean scale.

Do	Re	Mi	Fa	Sol	La	Si	Do
L	$\frac{8}{9}L$	$\frac{64}{81}L$	$\frac{3}{4}L$	$\frac{2}{3}L$	$\frac{16}{27}L$	$\frac{128}{243}L$	$\frac{1}{2}L$

Table 1: C major in the Pythagorean scale

To return to our problem, let's find the ratio between two adjacent notes. They are all $8/9$ except for the one between Mi and Fa and the one between Si and Do, which are $243/256$. If you actually calculate these ratios in decimals, you get

$$\frac{8}{9} = 0.8888\cdots, \qquad \frac{243}{256} = 0.9492\cdots \tag{11}$$

Thus, the difference between Mi and Fa and the one between Si and Do are indeed smaller than the difference between other adjacent ones. In other words, we indeed found out that the difference between the third tone (which is called "Mi") and the fourth tone (which is called "Fa") and the difference between the seventh tone (which is called "Si") and the eighth tone (which is called "Do") are smaller than the difference of other adjacent tones.

But, does it really mean that the tone difference of the former is exactly "half" of the tone difference of the latter? In other words, is the tone difference between Do and Re exactly the "double" of the tone difference between Mi and Fa? For this to be satisfied, the square of 243/256 must be 8/9. Let's check it. We have

$$\left(\frac{243}{256}\right)^2 = \frac{59049}{65536} \approx 0.9010 \tag{12}$$

which is close, but different from 8/9. Thus, strictly speaking, in the Pythagorean scale, the tone difference between Mi and Fa and the one between Si and Do are not *exactly* half, but *approximately* half the one between the tone difference between other adjacent ones.

Actually, this fact leads to a problem. To see what the problem is, let's find out what would correspond to Fa sharp. As it is one full tone higher than Mi, we have

$$\frac{64}{81}L \times \frac{8}{9} = \frac{512}{729}L \tag{13}$$

Then, what is the tone difference between Fa and Fa sharp? We have

$$\frac{512}{729}L \div \frac{3}{4}L = \frac{2048}{2187} = 0.9364\cdots \tag{14}$$

which is different from 243/256 in (11).

There is no problem in the Pythagorean scale if we use only C major, but sometimes we need to switch the code in the middle of a song. For example, the Pythagorean scale could be a problem for C sharp major, whose "Mi" is Fa, and whose "Fa" is Fa sharp. Let me explain what I mean. Earlier, we saw that the tone difference between Mi and Fa of a major needs to be 243/256, which is not satisfied by C sharp major. Thus, if we use the Pythagorean scale for C sharp major, it may not sound that harmonious.

What should we do? Now comes what is called "equal temperament." It is a universal standard in music since the 19th century. The idea is to make the semitone difference between Mi and Fa and Si and Do, *exactly* the half of the tone difference of other adjacent notes. First, notice that there are a total of 5 full tone differences and 2 semitone differences in one octave. If a full tone difference is two semitone differences, there are a total of 12 semitone difference in one octave. If we make each semitone difference equal, say, the ratio is x, we need to have

$$x^{12} = \frac{1}{2} \tag{15}$$

So, x is given by $(1/2)^{1/12}$. Thus, we have Table. 2.

Do	Re	Mi	Fa	Sol	La	Si	Do
L	$\left(\frac{1}{2}\right)^{2/12} L$	$\left(\frac{1}{2}\right)^{4/12} L$	$\left(\frac{1}{2}\right)^{5/12} L$	$\left(\frac{1}{2}\right)^{7/12} L$	$\left(\frac{1}{2}\right)^{9/12} L$	$\left(\frac{1}{2}\right)^{11/12} L$	$\frac{1}{2}L$

Table 2: C major in equal temperament

Now, as the tone differences (i.e., the ratio of the length of strings) are equal, we can play the song in any major. Nevertheless, there is a small disadvantage of equal temperament. Remember how we constructed the Pythagorean scale. Sol corresponds to $(2/3)L$, a simple ratio to L, which Do corresponds to. Now, Sol corresponds to

$$\left(\frac{1}{2}\right)^{7/12} L \approx 0.6674L \tag{16}$$

which is not equal to $(2/3)L \approx 0.6667L$. So, strictly speaking, Sol to Do is no longer a simple ratio. However, our ears are not good enough to notice the difference between 0.6667 and 0.6674. Therefore, equal temperament is good enough!

Final comment. In this article, we talked about the length of string that corresponds to a note, but we could instead talk about the frequency that corresponds to a note. Sound is a wave, and frequency is the number of oscillations of the wave in a second. It turns out that the length of string is inversely proportional to the frequency. Thus, a note an octave higher has double the frequency of the original note. Do is 261.63 Hz. In other words, the wave it produces oscillates 261.63 per second. Do, an octave higher is 523.26 Hz, the double of 261.63 Hz.