Domagala-Lewandowski-Meissner formula reproduces the Bekenstein-Hawking entropy

"Approximation of the naive black hole degeneracy" was my first published paper in a journal. Now, I think that the paper is somewhat useless except for the part where I proved that Domagala-Lewandowski-Meinssner formula reproduces Bekenstein-Hawking entropy, and the part where I showed that the black hole entropy is given by what I called "naive" black hole entropy.[1] In this article, I review the introduction and this part in a pedagogical manner.

1 Black hole entropy and the area spectrum

Even though what we are going to review in this section is already reviewed in our earlier article "Discrete area spectrum and the black hole entropy I," let's repeat it here to fix the notation. According to loop quantum gravity, there is something called unit areas. Let's say that we have the following unit areas.[2]

$$A_1, A_2, A_3, A_4, A_5, A_6 .... \tag{1}$$

Here, we used the notation that the ith unit area is A_i. Then, a generic area should be partial sum of them, including the case in which the same unit areas are repeated in the sum. In other words, a generic area has many partitions, each of which must be A_i for some i.

Given this, an interesting proposal was made by Rovelli in 1996. As black hole entropy is given by $A/4$, the degeneracy of black hole is given by $e^{A/4}$. Rovelli proposed that the black hole degeneracy is obtained by counting the number of ways in which the area of black hole can be expressed as the sum of unit areas. In other words, $N(A)$ the degeneracy of the black hole with area A is given by following.

$$N(A) := \left\{ (i_1, i_2, i_3 \cdots), \sum_x A_{i_x} = A \right\} \tag{2}$$

[1]The "naive" black hole entropy is the black hole entropy described in Section 1 without imposing any new condition. In traditional black hole entropy one imposes an extra condition after counting the degeneracy described in Section 1.

[2]Unit areas are given by the eigenvalues of area operator which turn out to be discrete. You will understand what eigenvalues are and why unit areas are given by the eigenvalues of area operator after you read our later article "A short introduction to quantum mechanics I: observables and eigenvalues."

Here, I want to note an important point. For the parenthesis in the above formula $(\cdots, a, \cdots, b, \cdots)$ should be regarded different from $(\cdots, b, \cdots, a, \cdots)$. In other words, the order in the summation is important. It is because area quanta on a black hole are distinguishable as they can have certain locations. If they have certain locations, we can distinguish one area quantum from the other.

2 Domagala-Lewandowski-Meissner trick

We already reviewed Domagala-Lewandowski-Meissner formula in our earlier article "Discrete area spectrum and the black hole entropy II." Here, we write it out mathematically. It is given as follows.

$$1 = \sum_{i=1}^{\infty} e^{-A_i/4} \tag{3}$$

3 Compositions

We already reviewed compositions in our earlier article. The key equation was

$$\sum_{m=1}^{\infty} (x + x^2 + x^3 + \cdots)^m = \frac{(x + x^2 + x^3 + \cdots)}{1 - (x + x^2 + x^3 + \cdots)} = \sum_{n=1}^{\infty} c(n) x^n \tag{4}$$

4 Our theorem

Now, let's apply the lesson from our earlier section to our case, namely, Bekenstein-Hawking entropy. The fact that $\{A_i, A_j\}$ should be regarded different from $\{A_j, A_i\}$ suggests that the calculation of black hole entropy has a similar structure to "compositions" in which the order is taken into account. Considering this, (2) can be translated into

$$\sum_{m=1}^{\infty} (e^{-sA_1} + e^{-sA_2} + \cdots)^m = \frac{e^{-sA_1} + e^{-sA_2} + \cdots}{1 - (e^{-sA_1} + e^{-sA_2} + \cdots)} = \sum_{A} N(A) e^{-sA} \tag{5}$$

where s is an arbitrary parameter. (If you are not sure how this is derived, take the steps again that led to (4).) It is easy to see that the above formula converges for s such that

$$e^{-sA_1} + e^{-sA_2} + \cdots < 1 \tag{6}$$

and diverges for s such that

$$e^{-sA_1} + e^{-sA_2} + \cdots \geq 1 \tag{7}$$

However, from Domagala-Lewandowski-Meissner formula (3), we have:

$$e^{-A_1/4} + e^{-A_2/4} + \cdots = 1 \tag{8}$$

Therefore, by examining (6) and (7), we can see that (5) converges for $s > \frac{1}{4}$, and diverges for $s \leq \frac{1}{4}$. Given this, if we closely examine the right-hand side of (5), the only conclusion that we can draw is that (8) implies

$$N(A) \sim P(A)e^{A/4} \tag{9}$$

for large enough A, and for $P(A)$ which does not increase or decrease faster than an exponential function.

Summary

- Domagala-Lewandowski-Meissner formula implies the degeneracy of black hole with area A is given by

  $$N(A) \sim P(A)e^{A/4}$$

  for large enough A, and for $P(A)$ which does not increase or decrease faster than an exponential function.