Neutrino oscillation, clarified

In an earlier article "Neutrino oscillation," I explained neutrino oscillation at a layman's level. However, I couldn't explain it fully as the knowledge of quantum mechanics was not assumed. In this article, I will explain neutrino oscillation using quantum mechanics.

Neutrino oscillation occurs because the flavor eigenstates of neutrinos are not equal to their mass eigenstates; they are related by unitary matrices as follows:

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} \tag{1}$$

where ν_e, ν_μ and ν_τ denote electron neutrino, muon neutrino, and tau neutrino respectively, and ν_1, ν_2 and ν_3 denote mass eigenstates and U denotes the matrix that relates them. This matrix is called the Pontecorvo-Maki-Nakagawa-Sakata matrix (PMNS matrix), Maki-Nakagawa-Sakata matrix (MNS matrix), lepton mixing matrix, or neutrino mixing matrix.

The above expression can be re-expressed as:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle \tag{2}$$

where $|\nu_\alpha\rangle$ denotes the flavor eigenstates, $|\nu_i\rangle$ the mass eigenstates, and $U_{\alpha i}$ the PMNS matrix.

Problem 1. Show that the PMNS matrix has to be a unitary matrix if both the flavor eigenstates and the mass eigenstates are properly normalized, i.e.,

$$\langle \nu_\alpha | \nu_\beta \rangle = \delta_{\alpha\beta}, \qquad \langle \nu_i | \nu_j \rangle = \delta_{ij} \tag{3}$$

Now, we will consider the case in which neutrinos are flying in free space (i.e. with the potential being zero) and see how the wave functions evolve. From "A short introduction to quantum mechanics VIII: the time-dependent Schrödinger equation," the wave function evolves as follows:

$$|\nu_i(t)\rangle = e^{-i(E_i t - \vec{p}_i \cdot \vec{x})} |\nu_i(t=0)\rangle \tag{4}$$

where we have used the natural units. Also, we know that $E_i^2 = m^2 + p_i^2$ where we have used the natural units again.

Given this, remember that the masses of neutrinos are very small. This implies:

$$p_i = \sqrt{E_i^2 - m^2} \approx E_i - \frac{m^2}{2E_i} \tag{5}$$

Also, it implies that the neutrinos travel at a speed almost that of light. Therefore, the distance traveled by neutrinos during the time interval t is t in natural units. Plugging this value and (5) to (4), we get:

$$|\nu_i(t)\rangle = e^{-im_i^2 t/2E_i} |\nu_i(t=0)\rangle = e^{-im_i^2 t/2E} |\nu_i(t=0)\rangle \tag{6}$$

where in the last step we have assumed that the energy of neutrinos from the same source is independent of its type and for convenience called the energy "E."

From the above formula, it is obvious that the number of neutrinos of a given mass eigenstate doesn't change as $\langle\nu_i(t)|\nu_i(t)\rangle = \langle\nu_i(0)|\nu_i(0)\rangle$. However, the same cannot be said for the number of neutrinos of a certain flavor. To see this, we first need to derive (**Problelm 2.** Hint[1]):

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i} |\nu_i(t)\rangle = \sum_i \sum_\beta U_{\alpha i} U^*_{\beta i} e^{-im_i^2 t/2E} |\nu_\beta(0)\rangle \tag{7}$$

As there is no guarantee that the transition amplitude from $|\nu_\alpha(0)\rangle$ to $|\nu_\alpha(t)\rangle$ (i.e. $\langle\nu_\alpha(0)|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i} U^*_{\alpha i} e^{-im_i^2 t/2E}$) is a pure phase, there is similarly no guarantee that the initial ν_α would remain unchanged to other flavors. Therefore, we see that the number of ν_α is not conserved for a generic PMNS matrix. It is also easy to see that the probability that neutrino changes its flavor (i.e. ν_β changes to ν_α for $\alpha \neq \beta$) is non-zero for a generic PMNS matrix, as there is no guarantee that $\langle\nu_\beta(0)|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i} U^*_{\beta i} e^{-im_i^2 t/2E}$ is zero. This is the crux of neutrino oscillation. The flavor of neutrino changes.

Problem 3. Show that there would be no neutrino oscillation if all three mass eigenvalues of neutrinos were same. (Hint[2])

Another way of seeing this is that in such a case the flavor eigenstate would be the mass eigenstate since eigenstates of such a mass matrix (i.e. a matrix that is a scalar multiple of an identity matrix) can be any state. Remember that the number of neutrinos of a given mass eigenstate doesn't change.

The discussion so far seems somewhat abstract without explicit further calculations. However, for three neutrino oscillation, such explicit calculations can be too complicated to briefly explain here. As a hypothetical case in which there exist only two neutrino flavors, we can catch all the qualitatively features of neutrino oscillation. We will now consider explicit calculations for such a case.

Let the 2-dimensional analog of (1) be given as follows:

$$\begin{bmatrix} \nu_\alpha \\ \nu_\beta \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \tag{8}$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a unitary matrix. Here, there is no reason why a, b, c and d must be real numbers, but we can always *choose* them to be real numbers, by doing global gauge transformations on $\nu_1, \nu_2, \nu_\alpha, \nu_\beta$.

Problem 4. From the fact that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a unitary matrix, with the assumption that a, b, c and d are real numbers, show that (Hint:[3])

$$d = a, \quad c = -b, \qquad a^2 + b^2 = 1 \tag{9}$$

[1]Use the unitarity of the PMNS matrix to show that (2) implies $|\nu_i(0)\rangle = \sum_\beta U^*_{\beta i} |\nu_\beta(0)\rangle$

[2]Use the fact that $e^{-im_i^2 t/2E}$ is independent of i in this case and $\sum_i U_{\alpha i} U^*_{\beta i} = \delta_{\alpha\beta}$.

[3]Use the fact that a unitary matrix U satisfies $U^\dagger = U^{-1}$. For the expression of the matrix inverse of 2×2 matrix, look up our earlier article "Matrix inverses."

Here, we see that the matrix is completely determined if a and b are determined. Now, recall that $a^2+b^2=1$ is a circle with radius 1, if a is on the x-axis, and b is on the y-axis. How can we parameterize an arbitrary point on a circle? By an angle. If this angle is the angle between the point and the x-axis, we have $a=\cos\theta$ and $b=\sin\theta$. Thus, (8) can be re-written as

$$\begin{bmatrix}\nu_\alpha\\ \nu_\beta\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix} \tag{10}$$

Here, θ is called "mixing angle."

Problem 5. If there is no mixing (i.e., the flavor state is the mass eigenstate) what is the mixing angle?

Now, we can show (**Problem 6.**) the probability amplitude by which ν_β will transform into ν_α is given by

$$\begin{aligned}P(\beta\to\alpha) &= |\langle\nu_\beta(0)|\nu_\alpha(t)\rangle|^2=\langle\nu_\beta(0)|\nu_\alpha(t)\rangle\,\langle\nu_\beta(0)|\nu_\alpha(t)\rangle^*\\ &= (\cos\theta\sin\theta)^2\left(2-2\cos\frac{(m_1^2-m_2^2)t}{2E}\right)\\ &= \sin^2(2\theta)\sin^2\left(\frac{(m_1^2-m_2^2)t}{4E}\right)\end{aligned} \tag{11}$$

Therefore, we see clearly why neutrino flavor changing is called neutrino oscillation. Rather than changing only in one direction, the flavor of neutrinos change back and forth as time elapses as is clear from the term $\sin^2(\frac{(m_1^2-m_2^2)t}{4E})$. Also, notice that we can only know the difference of the mass squared from the neutrino oscillation; we can find neither the mass itself nor even which ones are bigger.

Now, let's go back to our real world, which has three neutrino flavors instead of two. Unlike the hypothetical two neutrino flavor world, it turns out that, in general, we can not choose a PMNS matrix that has only real entries. (We will come back to this in the next article.) In addition to the three mixing angles (θ_{12}, θ_{13}, θ_{23}, where θ_{12} is the mixing angle between ν_1 and ν_2 and so on), a phase often denoted as $\delta_{\rm CP}$ must be introduced. If we denote $\cos\theta_{12}$ as c_{12} and $\sin\theta_{12}$ as s_{12} and so on, the PMNS matrix can be written as

$$\begin{bmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{bmatrix}\begin{bmatrix}c_{13} & 0 & s_{13}e^{-i\delta_{\rm CP}}\\ 0 & 1 & 0\\ -s_{13}e^{i\delta_{\rm CP}} & 0 & c_{13}\end{bmatrix}\begin{bmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{bmatrix} \tag{12}$$

The mixing angles are measured to be,

$$\begin{aligned}\sin^2(2\theta_{12}) &= 0.85\pm0.02\\ \sin^2(2\theta_{23}) &= 0.99\pm0.01\\ \sin^2(2\theta_{13}) &= 0.09\pm0.01\end{aligned} \tag{13}$$

and the CP-violating phase,

$$\delta_{\rm CP}=-60^{\circ+66}_{\ -138} \tag{14}$$

The differences of squared masses are measured to be,

$$\begin{aligned}|m_1^2-m_2^2| &= (7.5\pm0.2)\times10^{-5}\mathrm{eV}^2\\ |m_2^2-m_3^2| &= (2.5\pm0.1)\times10^{-3}\mathrm{eV}^2\end{aligned}$$

Final comments. δ_{CP} introduced in (12) is called the CP violating phase. Those of you who read our earlier articles on CP violation will know what CP violation means. It turns out that CP is violated in neutrino oscillations only if PMNS matrix cannot be written only in terms of real numbers. I cannot explain why if you don't know some quantum field theory, but in the next article, I will at least explain why a (generic) PMNS matrix with three flavors cannot be written only in terms of real numbers; of course, it could turn out in the future that all the entries in the PMNS matrix are real, which would mean no CP violation in neutrino oscillations, but we do not know yet whether neutrino oscillation violates CP as zero CP violating phase is not experimentally excluded in (14) Anyhow the three neutrino flavor case is different from the hypothetical two neutrino flavor case, in which the CP violation is theoretically forbidden, as the two flavor version of the PMNS matrix can be written purely in terms of real numbers. Anyhow, we will revisit them in the next article.

Summary

- Neutrino flavor oscillates (i.e. the number of neutrinos with a certain flavor changes) as the neutrino flavor eigenstate is different from the neutrino mass eigenstate. This phenomenon is due to quantum mechanical time evolution of phase. Of course, the number of neutrino with a definite mass doesn't change.
- The neutrino oscillation gives us only the information about the differences of squared masses of neutrinos, not the absolute value of the masses.