Neutrino decoupling

1 Introduction

The standard big bang theory predicts the "Cosmic Neutrino Background" just like the Cosmic Microwave Background radiation. In other words, just like the energy of photons in the background of our Universe follow the Planck distribution (i.e., Bose-Einstein distribution) with a certain temperature, the energy of neutrinos in the background of our Universe follow the Fermi-Dirac distribution with a certain temperature. In this article, we will calculate this temperature, which is different from the CMB temperature.

In our very early universe, electron neutrinos (i.e., a type of neutrino), anti electron neutrinos, electrons and positrons were in themal equilibrium thanks to the following frequent reactions, which are due to weak interactions.

$$\bar{\nu}_e + e^- \quad \leftrightarrow \quad \bar{\nu}_e + e^- \tag{1}$$

$$\nu_e + e^+ \quad \leftrightarrow \quad \nu_e + e^+ \tag{2}$$

where ν_e and $\bar{\nu}_e$ denote an electron neutrino and its anti-particle, and e^- adn e^+ denote an electron and its anti-particle, positron. (Other types of neutrinos, the muon neutrino (ν_μ) and the tau neutrino (ν_τ) do not interact directly with electrons, so they are left out in the above reactions.)

However, such reactions became rare as the temperature of our Universe dropped below 1 MeV (i.e., when our Universe was around 1 sec old). Neutrinos were not no longer in a thermal equilibrium with the other particles. This is known as "neutrino decoupling."

Nevertheless, the electrons, positrons and photons still remained in thermal equilibrium through following reactions.

$$e^- + e^+ \rightarrow 2\gamma \tag{3}$$

$$2\gamma \rightarrow e^- + e^+ \tag{4}$$

where γ denotes a gamma ray (i.e., photon).

However, as the temperature of our Universe dropped below 0.5 MeV, the mass of electron (or positron), the reaction rate for (4) began to fall down compared to (3); the photons were no longer energetic to form pairs of an electron and a positron. By the time that the temperature of our Universe was far below than 0.5 MeV, the reaction (4) (almost) completely stopped. So did (3), as most of the positrons were already anihilated by the electrons.

Notice that the reaction (3) produces energy. This energy heats photons, but not neutrinos, because the latter already lost the thermal contact with the former. So, we expect that the temperature of the Cosmic Neutrino Background is lower than the temperature of the Cosmic Microwave Background.

To calculate this temperature, we need to use the conservation of entropy.

2 The conservation of entropy

To derive the conservation of entropy, we need to first find an expression for the pressure of gas. Actually, we have already done so in two special limits, namely, in the non-relativistic limit, and in the ultra-relativistic limit. By recalling how we did so in these two cases, let's obtain a general formula.

Recall that, in non-relativistic limit, the contribution of the ith molecule with speed v_i and the mass m_i to the pressure is given by

$$P_i = \frac{\frac{1}{3}m_i v_i^2}{V} \tag{5}$$

where does this term come from? Recall that, we had $2m_i v_{ix}$ factor because the wall bounces the molecule, and its momentum transfer is $2m_i v_{ix}$. However, we had additional $v_{ix}/2$ factor, because it takes time for the molecule to come back to the original wall, which is proportional to $v_{ix}/2$. Then, the 1/3 factor is from the fact that $\overline{v_i^2} = \overline{v_{ix}^2} + \overline{v_{iy}^2} + \overline{v_{iz}^2} = 3\overline{v_{ix}^2}$. The only difference we have in the relativistic case is that the momentum transfer is $2\gamma m_i v_{ix}$. Thus,

$$P_i = \frac{(2\gamma m_i v_{ix})(v_{ix}/2)}{V} = \frac{\gamma m_i v_{ix}^2}{V} = \frac{\frac{1}{3}\gamma m_i v_i^2}{V} \tag{6}$$

Now, recall

$$v_i = \frac{\gamma m_i v_i}{\gamma m_i} = \frac{p_i}{E_i} \tag{7}$$

Thus, (6) becomes

$$P_i = \frac{p_i^2/3E_i}{V} \tag{8}$$

In conclusion, we obtain

$$P = \int \frac{gp^2 dp}{2\pi^2} f(E) \frac{p^2}{3E} \tag{9}$$

where

$$f(E) = \frac{1}{e^{(E-\mu)/T} \pm 1} \tag{10}$$

From now on, we will ignore again the chemical potential μ, which is valid in our case of interest. Then, the BE and the FD distribution is only a function of E/T.

Problem 1. Show that the following is satisfied if f is only a function of E/T. (Hint[1])

$$\frac{\partial f}{\partial T} = -\frac{E}{T}\frac{\partial f}{\partial E} \tag{11}$$

[1] Calculate $\partial f/\partial T$ and $\partial f/\partial E$ using chain rule, and compare them.

Problem 2. Derive the following relation by using (11) and integration by parts.

$$\frac{\partial P}{\partial T} = \frac{\rho + P}{T} \tag{12}$$

We derived the above formula for a gas of single component, but it is valid for a gas with multiple components, as long as they are in thermal equilibrium (i.e., $T_a = T$) because we have

$$\frac{\partial \sum_a P_a}{T} = \frac{\sum_a \rho_a + \sum_a P_a}{T} \tag{13}$$

where P_a and ρ_a is the pressure and the mass density of the ath component, which implies

$$\frac{\partial P}{\partial T} = \frac{\rho + P}{T} \tag{14}$$

where $P = \sum_a P_a$ and $\rho = \sum_a \rho_a$ are the total pressure and the total mass density.

Now, recall that, in our article on general relativity, we obtained

$$\frac{\partial \rho}{\partial t} = -3\frac{\dot{a}}{a}(\rho + P) \tag{15}$$

From this, we can derive

$$\frac{1}{a^3}\frac{\partial \rho a^3}{\partial t} = -3\frac{\dot{a}}{a}P \tag{16}$$

$$\frac{1}{a^3}\frac{\partial (\rho + P)a^3}{\partial t} - \frac{\partial P}{\partial t} = 0 \tag{17}$$

$$\frac{1}{a^3}\frac{\partial (\rho + P)a^3}{\partial t} - \frac{\partial T}{\partial t}\frac{\partial P}{\partial T} = 0 \tag{18}$$

$$\frac{1}{a^3}\frac{\partial (\rho + P)a^3}{\partial t} - \frac{\partial T}{\partial t}\frac{\rho + P}{T} = 0 \tag{19}$$

$$\frac{1}{a^3}\frac{\partial}{\partial t}\left(\frac{\rho + P}{T}a^3\right) = 0 \tag{20}$$

In the last article, we obtained that the entropy density is $s = (\rho + P)/T$. As the volume is proportional to a^3, we conclude that the total entropy is conserved.

3 The Cosmic Neutrion Background temperature

So, how can we use the conservation of entropy to calculate the temperature of the Cosmic Neutrino Background?

Let's calculate the total entropy of electrons, positrons and photons when they were in thermal equilibrium with neutrino. As mentioned, this is before electron-positron annihilation. Let's call the temperature and the scale factor then by T_n and a_n. Also, recall that $g = 2$ for photons, $g = 2$ for electrons (spin up and spin down) and $g = 2$ for positrons (spin up and spin down). In our earlier article, we have seen that the entropy density per degree of freedom of relativistic Fermi gas is 7/8 of the one of Bose gas. Thus,

$$S = sV = (2 + \frac{7}{8} \times 2 + \frac{7}{8} \times 2)\frac{2\pi^2}{45}T_n^3(2\pi^2 a_n^3) = \frac{11}{4}\frac{4\pi^4}{45}(T_n a_n)^3 \tag{21}$$

where we used the fact that the volume of a 3-sphere with radius r is $2\pi^2 r^3$. (Of course, only the fact that the volume is proportional to a^3 is important in actual comparison.) This value, the total entropy of electrons, positrons, and photons is preserved after the neutrino decoupling. We did not include the entropy of neutrinos in this calculation at the first place.

Now, let's calculate the total entropy of electrons, positrons and photons after electron-positron annhilation. It is given by the entropy of photons, because the number of electrons and positrons survived after the annhilation are negligible. Thus, we have

$$S = sV = 2\frac{2\pi^2}{45}T^3(2\pi^2 a^3) = 2\frac{4\pi^4}{45}(Ta)^3 \tag{22}$$

Equating (21) and (22), we have

$$T_n a_n = \left(\frac{4}{11}\right)^{1/3} Ta \tag{23}$$

So, what is T_ν, the temperature of the Cosmic Neutrino Background now? In our earlier article "The CMB today," we have obtained that Ta is a constant for the CMB. By taking a similar step, it is very easy to see that Ta for the Cosmic Neutrino Background (i.e., the neutrino temperature multiplied by the scale factor) is constant as well. Thus, we have $T_n a_n = T_\nu a$. Plugging this to (23), we get

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T \tag{24}$$

As the CMB temperature now is about 2.73 K, the Cosmic Neutrino background temperature now is about 1.95 K. However, as the Cosmic Neutrino Background has never been detected, so this temperature is not confirmed by observations yet.

Summary

- Neutrino decoupled before electron-positron annihilation. Therefore, the energy released during electron-positron annihilation was transferred to photons, but not to neutrinos. Therefore, the temperature of the Cosmic Microwave Background is higher than the temperature of the Cosmic Neutrino Background. The latter can be expressed in terms of the former by using the conservation of entropy.