Dimensions of orthogonal group and unitary group

In an earlier article, we have introduced orthogonal matrix $O(N)$ and special orthogonal matrix $SO(N)$. In the last article, you have shown that they form groups. In this article, we will calculate their dimensions using three methods. Then, we will calculate the dimension of unitary group $U(N)$ and special unitary group $SU(N)$.

First method. A, an $N \times N$ matrix, has N^2 entries. For A to be an orthogonal matrix, it needs to satisfy $AA^T = I$. Now, notice that any $N \times N$ matrix A satisfies

$$(AA^T)^T = AA^T \tag{1}$$

In other words, AA^T is a symmetric matrix.

Problem 1. Show that a symmetric $N \times N$ matrix has $N(N+1)/2$ independent components.

Thus, $AA^T = I$ gives $N(N+1)/2$ independent conditions. As A has N^2 entries, which satisfy $N(N+1)/2$ equations, there are $N^2 - N(N+1)/2 = N(N-1)/2$ independent degree of freedoms for $O(N)$. This is the dimension of $O(N)$. $O(N)$ group is $N(N-1)/2$ dimensional manifold. To calculate the dimension of $SO(N)$, notice that $O(N)$ has either determinant 1 or -1. Thus, $SO(N)$ is half of $O(N)$. As we know that cutting a manifold equally to two parts doesn't diminish the dimension, $SO(N)$ is also $N(N-1)/2$ dimensional manifold.

Second method. In earlier articles, we have seen that special orthogonal matrices correspond to rotation matrices; length is invariant under rotation, and $SO(N)$ preserves the length. To rotate something, we need to pick a plane that rotates. For example, if you rotate a point (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ along $2-4$ plane by θ we will have

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= x_2 \cos\theta - x_4 \sin\theta \\ x_3' &= x_3 \\ x_4' &= x_4 \cos\theta + x_2 \sin\theta \end{aligned} \tag{2}$$

There are total $\binom{4}{2} = 6$ number of two sets of plane that we can rotate. So, $SO(4)$ is 6 dimensional. In general, $SO(N)$ is $\binom{N}{2} = \frac{N(N-1)}{2}$ dimensional. Notice that $SO(3)$

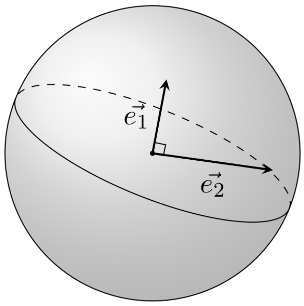

Figure 1: Choosing $\vec{e}_2$

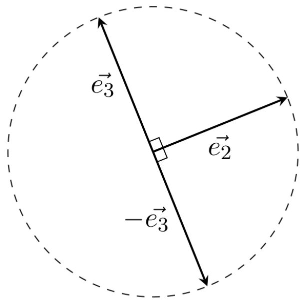

Figure 2: Choosing $\vec{e}_3$

(rotation in 3-d) is 3-dimensional, because $3(3-1)/2 = 3$. For $N \neq 3$, the dimension of $SO(N)$ is not equal to N.

Third method. This is the method that I made up myself when I first learned orthogonal matrix when I was a freshman in university. Of course, I am sure that I am not the one who first had this idea. Orthogonal matrix is equivalent to choose orthonormal basis. For example, if you write an $O(3)$ matrix as follows:

$$O = \left[\begin{array}{c:c:c} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array} \right] \tag{3}$$

Then, we have

$$O^T O = \left[\begin{array}{c} \vec{e}_1 \\ \hdashline \vec{e}_2 \\ \hdashline \vec{e}_3 \end{array} \right] \left[\begin{array}{c:c:c} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

In component notation, we have

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \tag{5}$$

$\vec{e}$s are indeed orthonormal basis. Now comes my idea. Let's choose $\vec{e}_1$ first. It's a vector with magnitude 1. Thus, it can point anywhere in 2-sphere. ($e_{1x}^2 + e_{1y}^2 + e_{1z}^2 = 1$.) Thus, there is 2 degree of freedom. Once, we chose $\vec{e}_1$, we can then choose $\vec{e}_2$. As e_2 has to be orthogonal to $\vec{e}_1$, $\vec{e}_2$ has to lie in 1-sphere (i.e. circle) orthogonal to e_1. See Fig.1. e_2 can lie anywhere in the dotted circle. Thus, there is 1 degree of freedom. Once we chose e_2, e_3 has to be orthogonal to both e_1 and e_2. The condition that it has to be orthogonal to e_1 forces it to lie in the dotted circle. The condition that it has to be orthogonal to e_2 forces it to be one of either of two points in the dotted circle see Fig. 2. So, there is no degree of freedom, but just two choices. One point will give the determinant of the matrix 1, and the other -1. Thus, the dimension of $SO(3)$ (as well as $O(3)$) is given by $2 + 1 = 3$.

In general, for $O(N)$ matrix, e_1 will lie in S^{N-1}. e_2 will lie in S^{N-2} and so on. e_N can have two choices: the one with determinant 1 and the other with determinant -1. Thus, the dimension of $O(N)$ is given by

$$(N-1)+(N-2)+\cdots+1+0=\frac{N(N-1)}{2} \tag{6}$$

Now, we turn to calculate the dimension of unitary group. To recall, a unitary group U satisfies $UU^\dagger = I$. $U(N)$ is an $N\times N$ unitary matrix.

Given this, we can take the similar step as the firth method to calculate the dimension of orthogonal group. A complex valued $N\times N$ matrix B has $2N^2$ real dimensions, as there are N^2 complex entries, and each complex number has 2 real dimensions (the real part, and the complex part).

For B to be a unitary matrix it needs to satisfy $BB^\dagger = I$. Now, notice that any complex valued $N\times N$ matrix B satisfies

$$(BB^\dagger)^\dagger = BB^\dagger \tag{7}$$

In other words, $BB^\dagger$ is a Hermitian matrix.

Problem 2. Show that a Hermitian $N\times N$ matrix has N^2 real independent components.

Thus, $BB^\dagger = I$ gives N^2 independent real conditions. As B has $2N^2$ independent real components, there are $2N^2 - N^2 = N^2$ independent real degree of freedoms for $U(N)$. This is the dimension of $U(N)$.

$SU(N)$ is a $N\times N$ unitary matrix with determinant 1. In case of $O(N)$, its determinant was 1 or -1. Thus, the dimension of $O(N)$ was the same as the one of $SO(N)$. However, in case of unitary group, we have

$$\det U \det U^\dagger = \det I \qquad \to \qquad \det U(\det U)^* = 1 \tag{8}$$

Thus, the determinant of a unitary group is a complex number whose magnitude is 1, i.e., it is a so-called pure phase, and can be expressed as

$$\det U = e^{i\phi} \tag{9}$$

for $0\le\phi<2\pi$. So, the determinant of U can be expressed by one real number, ϕ. Thus, compared with the unitary group, a special unitary group has the extra condition that $\phi = 0$. In other words, it has one less dimension that the unitary group. In conclusion, the real dimension of $SU(N)$ group is N^2-1.

Final comment. Groups such as $SO(N)$, $O(N)$, $SU(N)$ that are also manifold are called "Lie group" (pronounced "Lee group") named after the 19th century Norwegian mathematician Sophus Lie. He was the one who first came up with Lie group. Complete classification of Lie group was done by the French mathematician Cartan. Lie group theory was a concept that found no immediate application in physics when it was first developed, but now it is essential in particle physics. We will talk more about it in our later articles.

Summary

- The dimension of $SO(N)$ is $\binom{N}{2}$.
- The dimension of $SU(N)$ is $N^2 - 1$.