Parity in quantum mechanics

Parity is the operation that sends the coordinate $(x, y, z) \to (-x, -y, -z)$. It is often denoted by P. In other words, it sends wave function as follows

$$P|\psi(x, y, z)\rangle = |\psi(-x, -y, -z)\rangle \tag{1}$$

Of course, in 1-dimension, we have $P|\psi(x)\rangle = |\psi(-x)\rangle$. See Fig. 1 and Fig. 2.

Now, notice

$$\begin{aligned} P^2|\psi(x, y, z)\rangle &= P(P|\psi(x, y, z)\rangle) \\ &= P|\psi(-x, -y, -z)\rangle \\ &= |\psi(x, y, z)\rangle \end{aligned} \tag{2}$$

Thus, we get $P^2 = I$. (I is the identity matrix). As I has eigenvalue 1, we conclude that the eigenvalues of P are 1 and -1. We say that the state with the eigenvalue 1 has "even parity" and the state with the eigenvalue -1 has odd parity. In other words,

$$P|\psi_{\text{even}}\rangle = |\psi_{\text{even}}\rangle \tag{3}$$

$$P|\psi_{\text{odd}}\rangle = -|\psi_{\text{odd}}\rangle \tag{4}$$

See Fig. 3 and Fig. 4. for examples of the wave functions with even parity and odd parity. This terminology should be clear, if you know the terms "even function" and "odd function."

Now, suppose that a certain Lagrangian or Hamiltonian has the parity symmetry; it doesn't distinguish between left and right. Then, the parity is

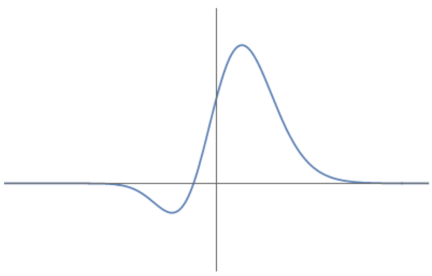

Figure 1: $|\psi(x)\rangle$

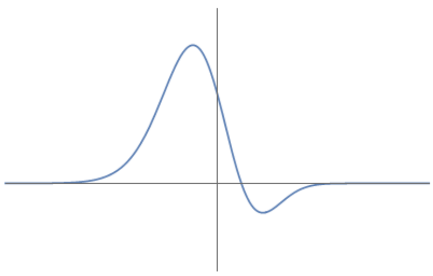

Figure 2: $P|\psi(x)\rangle$

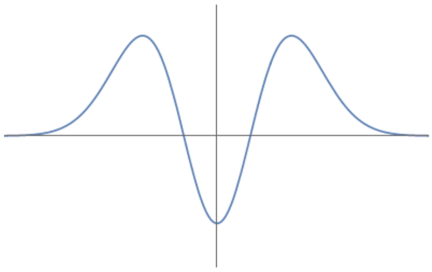

Figure 3: $|\psi_{\text{even}}\rangle$

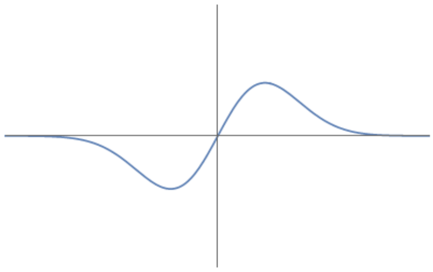

Figure 4: $|\psi_{\text{odd}}\rangle$

conserved, and we necessarily have $[P, H] = 0$. Given this, suppose there are no more than one eigenstate with a given eigenvalue of Hamiltonian. Then, we can show that such an eigenstate of Hamiltonian must be an eigenstate of parity as well. (**Problem 1.** Show this! Hint[1]) For example, the harmonic oscillator has the parity symmetry. The Hamiltonian is given by

$$H = \frac{-\hbar^2 \partial_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 \tag{5}$$

Upon the following parity transformation $x \to -x$,

$$H' = \frac{-\hbar^2 (-\partial_x)^2}{2m} + \frac{1}{2} m\omega^2 (-x)^2 = \frac{-\hbar^2 \partial_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 \tag{6}$$

the Hamiltonian remains the same. As we know that there are no more than one eigenstate with given eigenvalue of this Hamiltonian, all its eigenstates must have even parity or odd parity. Actually, the example in Fig. 3 is the eigenstate with energy eigenvalue $(2 + \frac{1}{2})\hbar\omega$ and the example in Fig. 4 is the eigenstate with energy eigenvalue $(1 + \frac{1}{2})\hbar\omega$.

In our earlier article "The symmetry of physical laws: the CPT theorem for laymen," we briefly mentioned that among the four elementary forces known to exist in our universe only weak force doesn't respect parity. Let me briefly mention how this was first discovered.

The following decays have been observed for the particles called τ^+ and θ^+.

$$\tau^+ \to \pi^+ + \pi^+ + \pi^- \tag{7}$$

$$\theta^+ \to \pi^+ + \pi^0 \tag{8}$$

π^+, π^0, π^- have electric charges $+e, 0, -e$ respectively, and they are called "pions." A pion has an odd parity regardless of its charge.

From the above decay channel, we can guess that the eigenvalue of the parity operator of the wave function of τ^+ is $(-1)^3$ while the one of θ^+ is

[1] Use $HP|\psi\rangle = PH|\psi\rangle$ and take the similar step as in the last article.

$(-1)^2$. (Note[2]) Therefore, τ^+ has an odd parity, as $(-1)^3 = -1$ and, θ^+ has an even parity, as $(-1)^2 = 1$.

The problem is that observed masses and lifetimes for τ^+ and θ^+ are same. So, it seems legitimate to suspect that they are actually the same particles. Otherwise, one needs to explain why nature allows two different particles that have all the same properties except for the parity and the decaying mode. This is known as $\tau - \theta$ puzzle.

The $\tau - \theta$ puzzle led two Chinese physicists Tsung-Dao Lee and Chen-Ning Yang at Columbia University in the United States to propose "parity doubling," which states that every particle comes with parity pairs: one with even parity, and one with odd parity. However, in hindsight, this was not the correct solution, and they came upon to raise another possibility that τ^+ and θ^+ are actually the same particle, and the weak interaction, responsible for the decays just mentioned, do not respect the parity conservation. Actually, they searched literature and concluded that there were no experimental evidences by then either to support or to reject parity conservation in weak interaction. Thus, in 1956, they suggested series of experiments that can check whether parity is conserved or not during weak interaction. In their paper, they concluded "one must perform an experiment to determine whether weak interactions differentiate the right from the left."[3]

Soon, Chien-Shing Wu, a female Chinese experimental nuclear physicist also at Columbia University, performed one of the experiments at the suggestion of Lee whom she personally knew. A cobalt-60 atom decays by emitting a beta particle (i.e. an electron). (This is called "beta decay" and weak interaction is responsible for it.) She and her collaborators aligned the spin (or the magnetic moment, which is parallel to spin) of cobalt-60 atoms using a strong magnetic field in a very cold environment; unless the atoms are cooled, the thermal agitation would hinder the atoms to be aligned. Then, they measured the direction in which the electrons are emitted. See Fig. 5. On the left, you see a cobalt-60 atom of which the spin is aligned along upward, and an electron being emitted along the dotted arrow direction. The angle between this direction and the direction of the spin alignment of the cobalt atom is θ. On the right, you see its mirror image. If a charged particle rotates clockwise, under its mirror image it would rotate anti-clockwise. Thus, the magnetic moment is reversed in its mirror image as drawn in the figure. However, notice that the right picture is the same thing as Fig. 6.

Thus, Wu and her collaborators wrote in their paper[4] "If an asymmetry

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[2] We need to take account the momentums of pions as well, but it turns out that they don't matter. More this on our later article "Historical Introduction to Weak Interaction."

[3] T. D. Lee and C. N. Yang (1956), Question of Parity Conservation in Weak Interactions, *Physical Review*, **104**, 254.

[4] Wu, C. S.; Ambler, E.; Hayward, R. W.; Hoppes, D. D.; Hudson, R. P. (1957). "Experimental Test of Parity Conservation in Beta Decay". *Physical Review*. **105** (4): 14131415

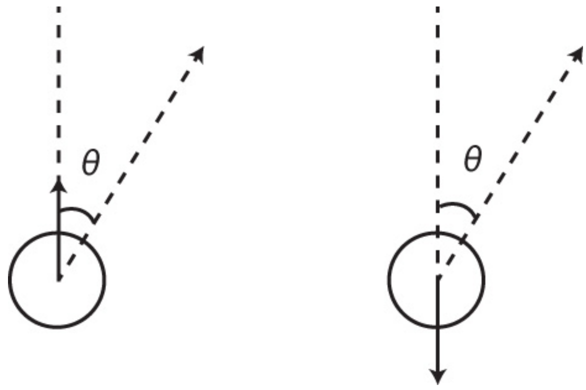

Figure 5: Co^{60} emitting an electron and its mirror image

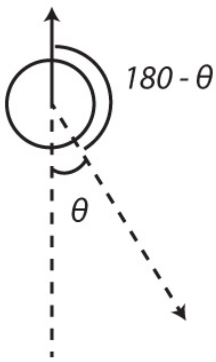

Figure 6: The equivalent picture of the right picture of Fig. 5

in the distribution between θ and $180^\circ - \theta$ (where θ is the angle between the orientation of the parent nuclei and the momentum of the electrons) is observed, it provides unequivocal proof that parity is not conserved in beta decay." They went on "This asymmetry effect has been observed in the case of oriented Co^{60}."

Therefore, in 1957, Lee and Yang were awarded Nobel Prize in Physics, but Wu wasn't.

Summary

- Parity is not conserved in weak force.