# Feynman path integral

In our earlier article on Ehrenfest theorem, we learned how Hamiltonian formulation of classical mechanics corresponds to quantum mechanics. In this article, we will see how Lagrangian picture of classical mechanics corresponds to quantum mechanics. This is due to Richard Feynman's Ph.D dissertation in 1948. This picture of quantum mechanics is called "Path integral formulation of quantum mechanics," and doesn't assume Schrödinger equation as input. Nevertheless, as we will see, it is equivalent to Schrödinger equation.

## 1 The sum over paths

Let's say that a particle is initially at a position $x_a$ when time is $t_a$. What is the probability that it will show up at a position $x_b$ at time $t_b$? To answer this question, first, notice that the wave function was initially $|x_a\rangle$. After the time $t_b - t_a$ has elapsed, the wave function becomes following:

$$e^{-iH(t_b-t_a)/\hbar}|x_a\rangle \tag{1}$$

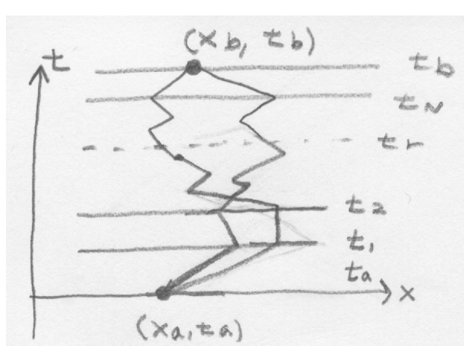
Given this, if we define the transition amplitude $U$ as follows,

$$U(x_a, t_a; x_b, t_b) = \langle x_b|e^{-iH(t_b-t_a)/\hbar}|x_a\rangle \tag{2}$$

the probability that we want is simply given by

$$U(x_a, t_a; x_b, t_b)U^*(x_a, t_a; x_b, t_b) \tag{3}$$

Therefore, $U$ indeed deserves the name "transition amplitude," since the above expression is transition probability. Notice also that $U$ is the representation of time evolution operator in position basis.



Now, to express $U$ in a slightly different way, let's define the Heisenberg picture position bra as follows:

$$|x_a, t_a\rangle = e^{iHt_a/\hbar}|x_a\rangle \tag{4}$$

(**Problem 1.** Prove that the above vector is an eigenvector of the Heisenberg picture position matrix with eigenvalue $x_a$ assuming that $|x_a\rangle$ is the eigenvector of the Schrödinger picture position matrix with eigenvalue $x_a$.) Then, it is easy to check that $U$ is given by:

$$U(x_a, t_a; x_b, t_b) = \langle x_b, t_b|x_a, t_a\rangle \tag{5}$$

Now, let's insert the following completeness relation to the above equation

$$1 = \int dx_1 |x_1, t_1\rangle\langle x_1, t_1| \tag{6}$$

then, we get:

$$U(x_a, t_a; x_b, t_b) = \int dx_1 \langle x_b, t_b|x_1, t_1\rangle\langle x_1, t_1|x_a, t_a\rangle \tag{7}$$

The interpretation of the above equation is follows. The particle is at $x_a$ when $t = t_a$ then it can move to any position $x_1$ when $t = t_1$ then finally arrive at $x_b$ when $t = t_b$. This suggests that $t_1$ that satisfies $t_a < t_1 < t_b$ is meaningful. Similarly, we can actually insert the completeness relation as many time as we want as follows:

$$\begin{aligned} U(x_a, t_a; x_b, t_b) &= \int dx_N \int dx_{N-1} \cdots \int dx_1 \langle x_b, t_b|x_N, t_N\rangle \\ &\times \end{aligned} \tag{8}$$

$$langlex_N, t_N|x_{N-1}, t_{N-1}\rangle \cdots \langle x_1, t_1|x_a, t_a\rangle \tag{9}$$

where $t_a < t_1 \cdots < t_{N-1} < t_N < t_b$. This represents the sum of all possible paths since the integration ranges for $dx$s are from the negative infinity to the positive infinity. See Fig.1.

## 2 Feynman path integral

In his Nobel lecture, Feynman described how he had come to discover the path integral formulation of quantum mechanics. As he struggled to find out a formulation of quantum mechanics based on action, at a beer party in a tavern in Princeton, he asked Prof. Jehle, a European, whether he had any idea. The next day, at the Princeton library, Jehle showed him the following equation in Dirac's paper, which in our notation is as follows:

$$\langle x', t+\epsilon|x, t\rangle \text{ is analogous to } \exp\left(i\int_t^{t+\epsilon} L(x, \frac{x'-x}{\epsilon})dt/\hbar\right) \tag{10}$$

Feynman then told Jehle that he guessed that Dirac meant "equal" by "analogous." Jehle objected, and commented that it would be useless to think about it. Feynman then did some calculation assuming that they were equal and derived Schrödinger's equation upon an extra assumption that they were not equal but proportional. Feynman showed his calculation to Jehle, who was very surprised and told him that it was an important discovery.

Now, let's find out what Feynman did. From (7), we have

$$U(x_a,0;x',t+\epsilon) = \int dx \langle x',t+\epsilon|x',t\rangle U(x_a,0;x,t) \tag{11}$$

Given this, we have to insert (10). As the Lagrangian is given as follows for infinitesimal $\epsilon$,

$$L(x,\frac{x'-x}{\epsilon}) = \frac{1}{2}m(\frac{x'-x}{\epsilon})^2 - V(x) \tag{12}$$

we have:

$$i\int_t^{t+\epsilon} L(x,\frac{x'-x}{\epsilon})/\hbar = \exp\left(\frac{i}{\hbar}\frac{m(x'-x)^2}{2\epsilon} - \frac{i}{\hbar}\epsilon V(x)\right) \tag{13}$$

Now, if we insert the proportionality constant $C(\epsilon)$ as follows, we get:

$$U(x_a,0;x',t+\epsilon) = C(\epsilon)\int dx \exp\left(\frac{i}{\hbar}\frac{m(x'-x)^2}{2\epsilon} - \frac{i}{\hbar}\epsilon V(x)\right) U(x_a,0;x,t) \tag{14}$$

As $x'$ cannot be far from $x$ when $\epsilon$ is small, we can Taylor-expand the above formula as follows:

$$U(x_a,0;x',t+\epsilon) = C(\epsilon)\int dx \exp\left(\frac{i}{\hbar}\frac{m(x'-x)^2}{2\epsilon}\right)(1-\frac{i}{\hbar}\epsilon V(x)) \tag{15}$$

$$\times \left(1+(x'-x)\frac{\partial}{\partial x} + \frac{1}{2}(x'-x)^2\frac{\partial^2}{\partial x^2}\right) U(x_a,0;x,t) \tag{16}$$

Recalling the following formulas:

$$\int d\xi e^{-A\xi^2} = \sqrt{\frac{\pi}{A}}, \quad \int d\xi \xi e^{-A\xi^2} = 0, \quad \int d\xi \xi^2 e^{-A\xi^2} = \frac{1}{2A}\sqrt{\frac{\pi}{A}} \tag{17}$$

we obtain:

$$U(x_a,0;x',t+\epsilon) = C(\epsilon)\sqrt{\frac{2\pi\hbar\epsilon}{-im}}\left(1-\frac{i\epsilon}{\hbar}V(x) + \frac{i\epsilon\hbar}{2m}\frac{\partial^2}{\partial x^2}\right) U(x_a,0;x,t) \tag{18}$$

The above equation must be satisfied when $\epsilon = 0$, this implies:

$$C(\epsilon) = \sqrt{\frac{m}{2\pi i\hbar\epsilon}} \tag{19}$$

Plugging this back in, and comparing the terms of order $\epsilon$ we get:

$$i\hbar\frac{\partial}{\partial t}U(x_a,0;x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right) U(x_a,0;x,t) \tag{20}$$

This is exactly Schrödinger's equation!

Summarizing, we have obtained:

$$\langle x',t+\epsilon|x,t\rangle = \sqrt{\frac{m}{2\pi i\hbar\epsilon}}\exp\left(i\int_t^{t+\epsilon} L(x,\frac{x'-x}{\epsilon})dt/\hbar\right) \tag{21}$$

Now, let's plug this back into (9). We get:

$$\langle x_b,t_b|x_a,t_a\rangle = \lim_{N\to\infty}\left(\frac{m(N+1)}{2\pi i\hbar(t_b-t_a)}\right)^{(N+1)/2}\int dx_N \int dx_{N-1}\cdots\int dx_1 \exp\left(i\int_{t_a}^{t_b} L(x,\dot{x})dt/\hbar\right) \tag{22}$$

(**Problem 2.** Prove this. Hint[1])

If we use the following notation:

$$\int \mathcal{D}\left[x(t)\right] = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{(N+1)/2}\int dx_N \int dx_{N-1}\cdots\int dx_1 \tag{23}$$

we get:

$$\langle x_b, t_b|x_a, t_a\rangle = \int_{x(t=t_a)=x_a}^{x(t=t_b)=x_b} \mathcal{D}\left[x(t)\right]\exp\left(i\int_{t_a}^{t_b} L(x,\dot{x})dt/\hbar\right) \tag{24}$$

Notice that the integration inside the exponent is for *all* paths that satisfy the boundary conditions $x(t=t_a)=x_a$ and $x(t=t_b)=x_b$.

Now, let us explain how the classical picture emerges from the path integral formulation of quantum mechanics. Consider a path $A$ that *doesn't* satisfy $\delta S=0$, and let's consider its neighboring paths $A+\epsilon\delta A$ for small $\epsilon$. Let's say their actions are given as follows

$$S(A+\epsilon\delta A) = S(A)+\epsilon\delta S \tag{25}$$

Remember that we are summing over all possible paths. As $\hbar$ is very small, the exponential in (24) (i.e. $e^{iS/\hbar}$) oscillates very rapidly, as you change $\epsilon$. This implies that their contributions to the amplitude have a tendency of getting canceled by neighboring paths; if you add up contributions that have random oscillating phases, you get zero. Thus, we see that the paths that are not the classical path contribute very little to the path integral.

However, this is not true for the path that satisfies $\delta S=0$, which is exactly the classical path. In this case, the action for its neighboring paths don't differ much, which makes their contribution to the amplitude not be canceled one another, because the exponentials in (24) don't oscillate much, but have somewhat similar values.

# 3 General case

In the last section, we have only considered the non-relativistic Newtonian case for the Lagrangian and the Hamiltonian,. In this section, we will see how it can be generalized. Also, we will use natural unit $\hbar=1$ for simplicity. For infinitesimal $\epsilon$, we have:

$$\begin{aligned}\langle x', t+\epsilon|x,t\rangle &= \langle x'|e^{-iH\epsilon}|x\rangle = e^{-iH\epsilon}\langle x'|x\rangle \\ &= e^{-iH\epsilon}\end{aligned} \tag{26}$$

$$langlex'|p\rangle\int dp\langle p|x\rangle = e^{-iH\epsilon}\int\frac{dp}{2\pi}e^{ip(x'-x)}$$

$$= \int e^{-iH\epsilon}\frac{dp}{2\pi}e^{ip\dot{x}\epsilon} = \int\frac{dp}{2\pi}e^{i(p\dot{x}-H)\epsilon} \tag{27}$$

Notice that the exponent is exactly Lagrangian. Plugging this back to (9), we get:

$$\begin{aligned}&\langle x_b, t_b|x_a, t_a\rangle \\ &= \lim_{N\to\infty}\int\frac{dp_N}{2\pi}\int\frac{dp_{N-1}}{2\pi}\cdots\int\frac{dp_1}{2\pi}\int dx_N\int dx_{N-1}\cdots\int dx_1\exp\left(i\int_{t_a}^{t_b}(p\dot{x}-H)dt/\hbar\right)\end{aligned}$$

[1] Set $\epsilon=(t_b-t_a)/(N+1)$ and also use $e^{a_1}e^{a_2}\cdots e^{a_n}=e^{a_1+a_2+\cdots a_n}$

which can be expressed more compactly as follows

$$\langle x_b, t_b | x_a, t_a \rangle = \int \mathcal{D}p\ \mathcal{D}x\ \exp\left( i \int_{t_a}^{t_b} (p\dot{x} - H) dt/\hbar \right) \tag{28}$$

under suitable normalizations of $\mathcal{D}p$ and $\mathcal{D}x$. Notice that the above result reproduces (24) where $L$ is given by (13) if $H = \frac{p^2}{2m} + V(x)$ as the $p$ integration is a Gaussian one upon completing the square, which only contributes to overall normalization factor.

# 4 Path integral in Euclidean space

Our world which we live in is not the Euclidean space but the Minkowski space. However. if we define the Euclidean time $\tau$ from Minkowskian time $t$ (i.e the usual time) by $\tau = it$ we would have Euclidean space, as the proper distance would be then given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 + (\Delta \tau)^2 \tag{29}$$

Contrary to what you may naively believe, Euclidean time turns out to be quite useful in physics. For example, in our later article "A Relatively Short Introduction to General Relativity," we will apply it to calculate the temperature of black holes. To this end, let's briefly review how it works.

Using (2), (24) can be re-expressed as

$$\langle x_b | e^{-iH(t_b - t_a)} | x_a \rangle = \int \mathcal{D}\left[x(t)\right] \exp\left( \int_{t_a}^{t_b} (\frac{1}{2} m (\frac{dx}{dt})^2 - V(x)) i\ dt \right) \tag{30}$$

which, upon Euclideanization, becomes

$$\langle x_b | e^{-H(\tau_b - \tau_a)} | x_a \rangle = \int \mathcal{D}\left[x(\tau)\right] \exp\left( \int_{\tau_a}^{\tau_b} - \left( \frac{1}{2} m (\frac{dx}{d\tau})^2 + V(x) \right) d\tau \right) \tag{31}$$

If $\beta = \tau_b - \tau_a$ and $x_a = x_b = x$ we have,

$$\mathrm{Tr}\left(e^{-\beta H}\right) = \int dx \langle x | e^{-\beta H} | x \rangle = \int \mathcal{D}\left[x(\tau)\right] \exp\left( \int_{\tau}^{\tau+\beta} - \left( \frac{1}{2} m (\frac{dx}{d\tau})^2 + V(x) \right) d\tau \right) \tag{32}$$

where $\mathcal{D}\left[x(\tau)\right]$ denotes all the curves that satisfy $x(\tau) = x(\tau + \beta)$ (i.e. curves with period $\beta$.) We see that the left-hand side is exactly the partition function in statistical mechanics. The right-hand side is also called the partition function.

## Summary

- Feynman path integral formalism is a Lagrangian formulation of quantum mechanics. It says that the quantum amplitude of an object from one position to another is given by the sum of amplitudes of all possible paths between these two positions along which the object can move.

- In particular,

$$\langle x_b, t_b | x_a, t_a \rangle = \int_{x(t=t_a)=x_a}^{x(t=t_b)=x_b} \mathcal{D}\left[x(t)\right] \exp\left( i \int_{t_a}^{t_b} L(x, \dot{x}) dt / \hbar \right)$$

- In the classical limit, only the classical path contributes (i.e., $\delta S$) to the path integral, as the contributions of the other paths cancel one another because the phase (i.e., $e^{iS/\hbar}$) vary rapidly from one path to the neighboring path.