Pauli matrices and spinor

In this article, we will represent the angular momentum operator by matrices. To this end, recall that in the last article we asserted that $L_+|j,\ m\rangle$ is proportional to $|j,\ m+1\rangle$ while $L_-|j,\ m\rangle$ is proportional to $|j,\ m-1\rangle$. Now, let's find the explicit proportionality constant. First, we will define $|j,\ m\rangle$s to be orthonormal as follows:

$$\langle j_1,\ m_1|j_2,\ m_2\rangle = \delta_{j_1 j_2}\delta_{m_1 m_2} \tag{1}$$

This is possible since $|j,\ m\rangle$s are eigenvectors of Hermitian matrices (i.e. L^2 and L_Z.). Now, we have:

$$\begin{aligned} L_+|j,\ m\rangle &= c|j,\ m+1\rangle \qquad (2) \\ \langle j,\ m|L_-L_+|j,\ m\rangle &= \langle j,\ m+1|c^*c|j,\ m+1\rangle \\ \langle j,\ m|L^2 - L_z^2 - \hbar L_z|j,\ m\rangle &= |c|^2 \\ (j(j+1) - m^2 - m)\hbar^2 &= |c|^2 \\ |c| &= \hbar\sqrt{j(j+1) - m(m+1)} \qquad (3) \end{aligned}$$

where from the first line to the second line, we used $(L_+)^\dagger = L_-$ and where from the second line to the third line, we used (18) in last article. Without loss of generality, one can choose c to be real. In other words, $c = |c|$. This is possible by performing global gauge transformation upon the eigenvector $|j,\ m+1\rangle$ in (2). Therefore, we conclude:

$$L_+|j,\ m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}|j,\ m+1\rangle \tag{4}$$

Similarly, one can show (**Problem 1.**)

$$L_-|j,\ m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}|j,\ m-1\rangle \tag{5}$$

Now, let's find the matrix representation of angular momentum, when $j = 1/2$. We know that its vector space is 2-dimensional since we have $|1/2,\ 1/2\rangle$ and $|1/2,\ -1/2\rangle$. We can represent this by:

$$|1/2,\ 1/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1/2,\ -1/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6}$$

The first one is called "spin up" and is the eigenvector of L_z with eigenvalue $\hbar/2$. The other one is called "spin down" and is the eigenvector of L_z with eigenvalue $-\hbar/2$. A general state in this vector space can be represented by a linear combination of these two vectors as follows:

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix} \tag{7}$$

This representation called "spinor" is used to express the state vector of spin-1/2 particle such as electron. Now, observe the followings:

$$L_z|1/2,\ 1/2\rangle = \frac{\hbar}{2}|1/2,\ 1/2\rangle, \qquad L_z|1/2,\ -1/2\rangle = -\frac{\hbar}{2}|1/2,\ -1/2\rangle \tag{8}$$

implies:

$$L_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

Similarly,

$$L_+|1/2,\ 1/2\rangle = 0, \qquad L_+|1/2,\ -1/2\rangle = \hbar|1/2, 1/2\rangle \tag{10}$$

implies:

$$L_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{11}$$

$$L_-|1/2,\ 1/2\rangle = \hbar|1/2, -1/2\rangle, \qquad L_-|1/2,\ -1/2\rangle = 0 \tag{12}$$

implies:

$$L_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{13}$$

Using, $L_x = (L_+ + L_-)/2$, and $L_y = (L_+ - L_-)/(2i)$, we obtain

$$L_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad L_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{14}$$

If we define σ by $\vec{L} = (\hbar/2)\vec{\sigma}$, we conclude:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{15}$$

These are called "Pauli matrices."

Recall that (6) are eigenvectors of L_z. What are the eigenvectors of L_x and L_y? Using (14), one can check that

$$\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \beta_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{16}$$

are eigenvectors of L_x with eigenvalues $\hbar/2$, $-\hbar/2$ respectively, and

$$\alpha_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \beta_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{17}$$

are eigenvectors of L_x with eigenvalues $\hbar/2$, $-\hbar/2$ respectively.

All the constructions in this article were based on 3 spatial dimensional cases; we used x, y, z and p_x, p_y, p_z. We showed that this led to an object that has two components. This is called a "spinor." Then, what would be the analogous relativistic construction? There, we would need x, y, z, t and p_x, p_y, p_z, E. Now, remember why we have 3 components of angular momentum in 3-dimensional space. We need to choose two spatial directions to rotate something on the plane spanned by these two spatial directions. As $\binom{3}{2} = 3$ a angular momentum in 3-dimensional space has 3 components. In 4-d spacetime dimension, $\binom{4}{2} = 6$. So, there are 6 ways to "rotate" an object in 4-d spacetime. 3 of them is related to the angular moomentum and 3 is related to the boost.

Wolfgang Pauli, who discovered Pauli matrices, knew that he had to extend his construction of Pauli matrices to a relativistic one, which would require 6 "relativistic" Pauli matrices. However, that turned out to be so difficult that he had to give up.

In 1928, Dirac found the solution. He constructed what is now called "Dirac equation," a relativistic version of Schrödinger equation for an electron. As by-products it all naturally followed that an electron has to have spin 1/2 and that orbital angular momentum of an electron is not conserved on its own, but only the sum of its orbital angular momentum and its spin angular momentum is conserved. It also followed that the g-factor had to be 2, as we mentioned in "Electron magnetic moment."[1]

Let me digress a little bit. In this article, we have seen that the wave function of an electron has two components as in (7). Dirac showed that, in 4d, it has four components. It turned out that two of the four components correspond to the wave function of electron, and the other two correspond to the wave function of positron, the anti-particle of electron. As we mentioned in "Charge conjugation," Dirac's prediction of positron is verified through its experimental discovery.

Dirac claimed that the knowledge of Pauli matrices did not help him at all in discovering Dirac's equation and spin angular momentum of electron, but the Japanese physicist Shinichiro Tomonaga wrote in his book "The story of spin" that he doubted it. Even though the knowledge of Pauli matrices coudn't have been directly helpful, it could be helpful in his contructrion of what is now called "Dirac matrices" which could be roughly described as 4-dimensional ("4d") analogs of Pauli matrices. As an aside, Pauli matrices are 2×2 matrices while Dirac matrices are 4×4. In 10d or 11d, in which string theory and M-theory live respectively, the Dirac matrices are 32×32, and a spinor has 32 components. All this would be interesting to talk about more in details, but this is out of scope for this series unfortunately.

Finally, let me conclude this article with a comment. One can take similar steps as in $j = 1/2$ case to calculate the angular momentum matrices for other j. (Of course, we are talking about 3d case as before.) For example, for $j = 1$, we obtain:

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{18}$$

[1]As mentioned there, it is not exactly 2, but only approximately 2.

$$L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{19}$$

$$L_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{20}$$

Problem 2. Suppose, an electron is in an eigenstate of L_z with eigenvalue $-\hbar/2$. What is the probability that its L_x will be $\hbar/2$ if you measure it?

Problem 3. Check (18), (19) and (20).

Problem 4. So far we have considered the angular momentum along x, y or z axis. More generally, we can consider the angular momentum along an arbitrary direction $\hat{r}$ given as follows using spherical coordinate system:

$$\hat{r} = \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k} \tag{21}$$

Show that the eigenspinors of angular momentum along $\hat{r}$ with eigenvalues respectively $\hbar/2$ and $-\hbar/2$ are given as follows:

$$\alpha_{\hat{r}} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}, \quad \beta_{\hat{r}} = \begin{pmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \tag{22}$$

For example, for L_z we have $\theta = 0$ and $\phi = 0$, so we obtain (6). For L_x, we have $\theta = \pi/2$ and $\phi = 0$, so we obtain (16). For L_y, we have $\theta = \pi/2$ and $\phi = \pi/2$, so we obtain (17) upto overall phase. (If $\vec{v}$ is a normalized eigenvector, $e^{i\lambda}\vec{v}$ is also a normalized eigenvector with the same eigenvalue. Here $e^{i\lambda}$ is the overall phase with λ being real.)(Hint[2])

Summary

- $|1/2,\ 1/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1/2,\ -1/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- The first one is called "spin up" and the other one is called "spin down."
- These two-dimensional space is called "spinor."
- $\vec{L} = (\hbar/2)\vec{\sigma}. \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

[2]The angular momentum along the direcion $\hat{r}$ is given by $\hat{L}\cdot\hat{r}$ where $\hat{L} = L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k}$.