Pentagon, the Fibonacci sequence, and the golden ratio

1 Pentagon and the golden ratio

See Fig. 1. $ABCDE$ is a pentagon. Its sides have the same length and the angles of the pentagon are all the same (i.e., $\angle EAB = \angle ABC = \angle BCD = \angle CDE = \angle DEA$). If you connect all the vertices in the pentagon one another, you get a "star" shape.

Problem 1. Calculate the value of these angles. (Hint[1])

Problem 2. Obtain $\angle ABE$ (or $\angle AEB$, which is the same). (Hint[2])

Of course, $\angle ABE = \angle CBD$

Problem 3. Obtain $\angle EBD$. (Hint[3])

Problem 4. Show $\triangle ABE$ is congruent to $\triangle UBE$. Therefore, $\overline{AB} = \overline{BU} = \overline{UE} = \overline{EA}$. Show also $\overline{BE} = \overline{BD}$.

Problem 5. Let's say $\overline{AB} = 1$, and $\overline{BE} = \phi$. Then, show that $\overline{UD} = \overline{UC} = \phi - 1$.

Problem 6. Show $\triangle UEB$ is similar to $\triangle UDC$.

Problem 7. Thus, show that

$$\phi(\phi - 1) = 1 \tag{1}$$

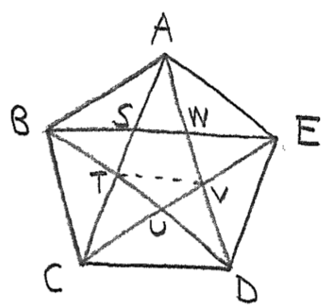

Figure 1: Pentagon

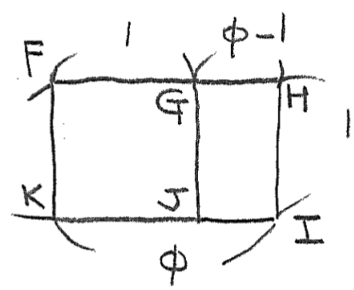

Figure 2: the golden ratio

[1]Figure out first what is the sum of all these angles. Then, you can divide it by 5 to obtain the angle. To figure out the sum, notice that a pentagon can be decomposed into three triangles. For example, $\triangle ABE$, $\triangle BCE$, $\triangle CDE$. The sum of the angles in the pentagon is equal to the sum of the angles in the three triangles.

[2]$\triangle ABE$ is an isosceles triangle.

[3]$\angle ABC = \angle ABE + \angle EBD + \angle CBD$.

Problem 8. Solve this equation to show that

$$\phi = \frac{1+\sqrt{5}}{2} = 1.61803\cdots \tag{2}$$

ϕ is called the "golden ratio." We have just seen that

$$\phi = \frac{1}{\phi - 1} = \frac{\overline{BE}}{\overline{AB}} = \frac{\overline{BE}}{\overline{ED}} = \frac{\overline{BE}}{\overline{AB}} \tag{3}$$

Similarly, by the similarity of triangles, it is straightforward to show that

$$\phi = \frac{1}{\phi - 1} = \frac{\overline{BS}}{\overline{ST}} = \frac{\overline{TV}}{\overline{SW}} \tag{4}$$

Another interpretation of the golden ratio would be following. See Fig. 2. You see three quadrangles. $\square FGKJ$, which is a square and $\square FHIK$ and $\square HIJG$.

Problem 9. Let's say $\square GHIJ$ is similar to $\square HIKF$. Then show that

$$\phi = \frac{\overline{KI}}{\overline{FK}} = \frac{\overline{HI}}{\overline{GH}} \tag{5}$$

Problem 10. By considering the isoscles triangle $\triangle BED$, show

$$\cos 72^\circ = \frac{1-\sqrt{5}}{4} \tag{6}$$

2 Fibonacci sequence

Consider the Fibonacci sequence defined by the following rules:

$$F_1 = 1, \quad F_2 = 1, \quad F_n + F_{n+1} = F_{n+2} \tag{7}$$

Let's use these rules to determine this sequence:

$$\begin{aligned}
1+1 &= 2 = F_3 \\
1+2 &= 3 = F_4 \\
2+3 &= 5 = F_5 \\
3+5 &= 8 = F_6 \\
5+8 &= 13 = F_7 \\
8+13 &= 21 = F_8 \\
13+21 &= 34 = F_9 \\
21+34 &= 55 = F_{10}
\end{aligned} \tag{8}$$

and so on. Given this, let's now calulate the ratio F_{n+1}/F_n and see what happens

$$\begin{gathered}
\frac{1}{1} = 1, \quad \frac{2}{1} = 2, \quad \frac{3}{2} = 1.5, \quad \frac{5}{3} = 1.66\cdots, \quad \frac{8}{5} = 1.6 \\
\frac{13}{8} = 1.625, \quad \frac{21}{13} = 1.6153\cdots, \quad \frac{34}{21} = 1.6190\cdots, \quad \frac{55}{34} = 1.6176\cdots
\end{gathered} \tag{9}$$

Do you notice something? The ratio approaches the golden ratio. Let's figure out why. Suppose the ratio approaches a certain constant. Then, we have

$$\frac{F_{n+1}}{F_n} = \frac{F_{n+2}}{F_{n+1}} \tag{10}$$

for very big n. (Strictly speaking, the equal sign in the above equation should be the approximation sign $\approx$.) Now, let's use $F_n + F_{n+1} = F_{n+2}$. Then,

$$\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n+1}}{F_{n+1}} = \frac{F_n}{F_{n+1}} + 1 \tag{11}$$

For convenience, let's define $r \equiv F_{n+1}/F_n$. Then, we have

$$\begin{aligned} r &= \frac{1}{r} + 1 && (12) \\ r - 1 &= \frac{1}{r} && (13) \\ r(r-1) &= 1 && (14) \end{aligned}$$

which is exactly (1). Thus, we see the reason why F_{n+1}/F_n approaches the golden ratio.

Now, another property of the Fibonacci sequence. Let's compare F_n^2 and $F_{n-1}F_{n+1}$.

$$\begin{aligned} &F_2^2 = 1, \qquad F_1F_3 = 2 \\ &F_3^2 = 4, \qquad F_2F_4 = 3 \\ &F_4^2 = 9, \qquad F_3F_5 = 10 \\ &F_5^2 = 25, \qquad F_4F_6 = 24 \\ &F_6^2 = 64, \qquad F_5F_7 = 65 \\ &F_7^2 = 169, \qquad F_6F_8 = 168 && (15) \\ & && (16) \end{aligned}$$

Do you see the pattern? We find the pattern that F_n^2 is $F_{n-1}F_{n+1} + 1$, when n is odd, $F_{n-1}F_{n+1} - 1$, when n is even. In other words,

$$F_n^2 = F_{n-1}F_{n+1} - (-1)^n \tag{17}$$

Will this pattern continue for bigger n? Once we prove the above relation we can be certain.

Problem 11. Prove (17) by induction.

It is possible to obtain a general formula for the Fibonacci sequence. First, notice that for a very large n, we have

$$\frac{F_{n+1}}{F_n} \approx \phi \tag{18}$$

which implies

$$F_n \approx c\phi^n \tag{19}$$

for some n. (**Problem 12.** Show (19) satisfies $F_n + F_{n+1} = F_{n+2}$.)

Of course, we know that (19) is only an approximate formula. Now, notice why (19) satisfies $F_n + F_{n+1} = F_{n+2}$. As you showed in Problem 11, it is because ϕ satisfies (1). Now, notice that (1) is a quadratic equation, which has two solutions. Let's call the other solution ϕ'. Then, it is easy to see that

$$F_n = d\phi'^n \tag{20}$$

also satisfies $F_n + F_{n+1} = F_{n+2}$, for the same reason as (19) satisfies this relation. Given this, it is now easy to check that the sum of (19) and (20) satisfies $F_n + F_{n+1} = F_{n+2}$. In other words,

$$F_n = c\phi^n + d\phi'^n \tag{21}$$

satisfies $F_n + F_{n+1} = F_{n+2}$. It implies that it has the possibility of being the Fibonacci sequence, which is actually the case, as we will see.

So, let's find ϕ'. It is given by

$$\phi' = \frac{1-\sqrt{5}}{2} \tag{22}$$

Problem 13. Show

$$\phi' = 1 - \phi = -\frac{1}{\phi} \tag{23}$$

Now, all we are left to do is finding c and d.

Problem 14. From $F_n + F_{n+1} = F_{n+2}$, show $F_0 = 0$. Then, by plugging $n = 0$ to (21). Show $d = -c$. Then, from $F_1 = 1$, show that

$$c = \frac{1}{\sqrt{5}}, \quad d = -\frac{1}{\sqrt{5}} \tag{24}$$

Thus, we finally obtain

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \tag{25}$$

So, why does F_{n+1}/F_n approach $(1+\sqrt{5})/2$ despite the presence of the term $((1-\sqrt{5})/2)^n/\sqrt{5}$?

If you actually calculate the value of ϕ', it is given by

$$\phi' = \frac{1-\sqrt{5}}{2} = -0.61803\cdots \tag{26}$$

Now, notice that ϕ'^n approaches fast 0, as n gets bigger and bigger. For example,

$$\phi'^5 \approx -0.09, \qquad \phi'^{10} \approx 0.008 \qquad \phi'^{25} \approx -6 \times 10^{-6} \tag{27}$$

As the last term in (25) is negligible for bigger and bigger n, we have (19), which makes F_{n+1}/F_n approach ϕ.

Final comment. The golden ratio and the Fibonacci sequence appear in the nature. I will update this article later, after reading some materials.

Summary

- The golden ratio ϕ is defined by $\phi = 1/(\phi - 1)$.
- The golden ratio appears in the ratio of some of the line segments in pentagon and the "star" in it.
- The Fibonacci sequence is defined by $F_n + F_{n+1} = F_{n+2}$.
- The ratio F_{n+1}/F_n approaches the golden ratio for large n.