Planet's motion around the Sun

In our earlier article, we obtained that the total energy of planet is given as follows in polar coordinate:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r} \tag{1}$$

We also saw that the angular momentum is constant as follows.

$$L = mr^2\dot{\theta} \tag{2}$$

In other words, $\dot{\theta} = L/(mr^2)$. If we plug this to (1), we obtain:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{3}$$

Therefore, by taking account the degree of freedom in the angular direction, we obtain an "effective potential" for radial direction as follows:

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{4}$$

In other words, the effective potential doesn't contain any θ or $\dot{\theta}$ dependence, and the total energy (3) can be expressed as follows:

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r) \tag{5}$$

Now, let's graph $U_{\text{eff}}(r)$. See Fig. 1. The coefficients are chosen arbitrary. Here, we see that the minimum of the effective potential is obtained when $r = 2$ with the value $U_{\text{eff}}(2) = -0.25$. If the energy of the planet happens to be -0.25, it will sit right there at $r = 2$ and the distance from the Sun will not change. In such a case, the orbit will be circular. However, if it has a little bit higher energy, but not more than zero, the distance from the Sun will oscillate. For example, if E is -0.2, the distance from the Sun will oscillate between around 1.4 and 3.6, as $U_{\text{eff}}(1.4) \approx U_{\text{eff}}(3.6) \approx 0$. In such a case, the orbit will be ellipse. However, if E is positive, the planet can move up to infinity since U_{eff} reaches only zero at infinity; the planet has still extra kinetic energy to move radially. In such a case, the orbit is hyperbola. When E is zero, the planet can move "just" up to infinity. In such a case, the orbit if parabola, as the boundary between ellipse and hyperbola is parabola.

Given this, using an approximation, we will show that the orbit of the planets around the Sun closes. If it doesn't close it will move like Fig. 2. We call this phenomenon "precession of perihelion." Perihelion is the point in which a planet is closest to the Sun. In Fig. 2 we see

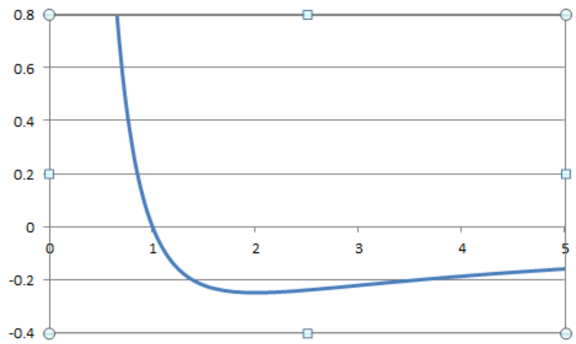

Figure 1: Effective potential $U_{\text{eff}}(r)$

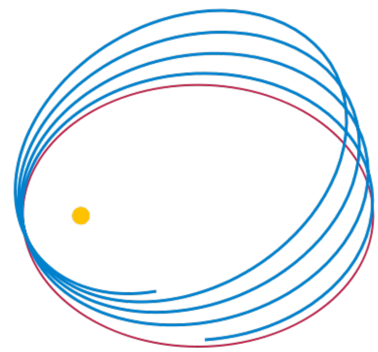

Figure 2: Precession of perihelion

that the perihelion namely precesses. Of course, we cannot call such an orbit ellipse, since an ellipse is a closed orbit.

By approximation, I mean that we will only consider the case that the orbit is nearly circular. In other words, when the planet is around the lowest point (i.e. stable equilibrium) of the effective potential. To this end, we can obtain "effective" force from effective potential as follows:

$$F_{\text{eff}}(r) = -\frac{\partial U_{\text{eff}}}{\partial r} = \frac{L^2}{mr^3} - \frac{GMm}{r^2} \tag{6}$$

Now, let's interpret the above formula. Using (2), we have:

$$F_{\text{eff}} = mr\dot{\theta}^2 - \frac{GMm}{r^2} \tag{7}$$

The first term on the right-hand side is centrifugal force and the second term Newton's universal gravitational force. We have the negative sign for this term, since it is directed inward (i.e. to the Sun). (i.e., in the direction r is decreasing)

Back to our original problem, if we call R the equilibrium position, (6) is zero when

$$L^2 = GMm^2R \tag{8}$$

If your position slightly deviates from this point, you will feel a force going back to this point just like an object connected to a spring. Given this, we will reduce our problem to that of harmonic oscillator. To this end, let's differentiate (6) around $r = R$, we get:

$$\frac{\partial F_{\text{eff}}}{\partial r} = -\frac{3L^2}{mr^4} + \frac{2GMm}{r^3} = -\frac{3GMmR}{r^4} + \frac{2GMm}{r^3} \tag{9}$$

$$\frac{\partial F_{\text{eff}}}{\partial r}(r = R) = -\frac{GMm}{R^3} \tag{10}$$

Therefore, if we Taylor-expand (6) around R, we have:

$$F_{\text{eff}}(r) = F_{\text{eff}}(R) + \frac{\partial F_{\text{eff}}}{\partial r}(r - R) + \cdot \tag{11}$$

$$= -\frac{GMm}{R^3}(r - R) + \cdots \tag{12}$$

Now, if we define $x \equiv r - R$, we have:

$$F_{\text{eff}} = -\frac{GMm}{R^3}x + \cdots \tag{13}$$

Therefore, for x small, in which we can ignore higher-order terms denoted by $\cdots$, this is exactly harmonic oscillator problem if we identify $k = GMm/R^3$. Now, using (9) in "Harmonic oscillator," we conclude that the period is given by:

$$T = 2\pi\sqrt{\frac{R^3}{GM}} \tag{14}$$

In other words, perihelion comes to perihelion after this amount of time. However, as this is exactly same as (4) in "Newton's law of universal gravitation and Kepler's third law," perihelion comes to perihelion after one rotation, making the orbit close.

In this article, we used approximation by Taylor-expanding, but actually it is also the case that one gets the same result even if one doesn't use approximation. However, that is out of scope for this article. This is left for the next article.

Problem 1. Redo the exercise done in this article, for a hypothetical case that the force between a planet and the Sun is inversely proportional to r^4. What is the period of the orbit? What can you conclude about its orbit?

Problem 2. In the case in which the mass of planet is not negligible compared to the mass of Sun how is (3) changed? Use the reduced mass.

Summary

- $m\ddot{r}$ for the motion of planet is given by the sum of centrifugal force and the gravitational force.
- The angular velocity dependence of the centrifugal force can be removed by expressing the centrifugal force in terms of angular momentum which is always constant. Then, the centrifugal force only depends on r.
- By combining the so-obtained centrifugal force and the gravitational force, we obtain the radial effective force.
- We can then examine the planet's motion around the equilibrium of the radial effective force by using linear approximation. By this way, we can find the period.