Polar coordinate and trigonometric functions defined over all values

On a two-dimensional plane, we need two numbers to locate a given point. See Fig. 1. The x-coordinate tells you how much you are displaced from the origin in x-direction, and the y-coordinate tells you how much you are displaced from the origin in y-direction. This coordinate system is called the Cartesian coordinate.

A natural question one may ask is: Is there another way of designating the position of a given point? Of course, the answer is yes. An example of such choice is "polar coordinate."

Instead of the displacements in x and y directions from the origin, we can measure the distance from the origin and in which direction it is located. See Fig. 2. "r" is the distance from the origin and θ is the angle that the line from the origin and the given point makes with x-axis.

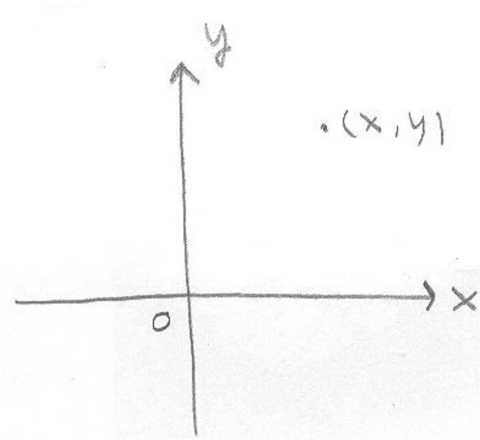

Figure 1: the Cartesian coordinate

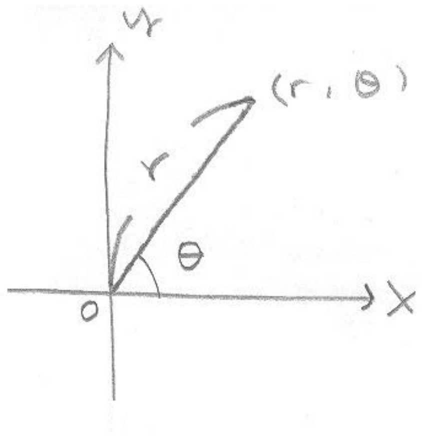

Figure 2: the polar coordinate

Applying the concept of the trigonometric functions introduced in the previous article, we can easily see that the following relations must hold between the polar coordinate and the Cartesian coordinate.

$$x = r\cos\theta \tag{1}$$

$$y = r\sin\theta \tag{2}$$

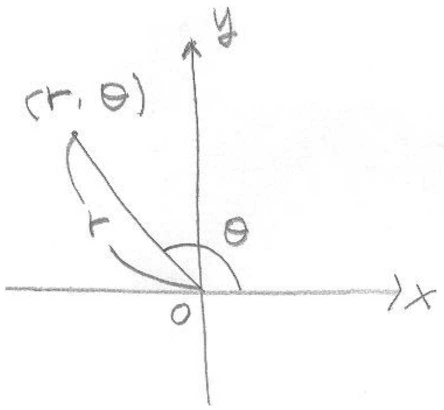

Figure 3: $\theta > 90°$

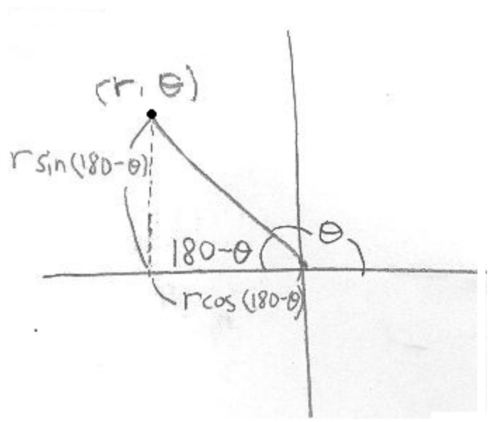

Figure 4: $90° < \theta < 180°$

which implies

$$r = \sqrt{x^2 + y^2}, \qquad \tan\theta = \frac{y}{x} \tag{3}$$

However, (1) and (2) may be a little bit unsettling. If you recall what you have learned in the previous article, the trigonometric functions are defined only for the value between 0 and 90°. You cannot have a right triangle one of whose angle is bigger than 90°. However, in some cases, θ is bigger than 90 degrees as in Fig. 3.

The trick is that we can use the above relations between the polar coordinate and the Cartesian coordinate as the definition of trigonometric functions for angles outside the range of 0 and 90°. For example, let's calculate the value for $\cos\theta$, and $\sin\theta$ when θ is between 90° and 180°. More precisely, let's express these values in terms of trigonometric functions of an angle between 0 and 90°, which we are already familar with. See Fig. 4. $180° - \theta$ is smaller than 90°. As before, the location of the point designated is given by (1) and (2).

By being careful of the fact that x is negative in this case, you can also see from the figure that

$$x = -r\cos(180° - \theta) \tag{4}$$
$$y = r\sin(180° - \theta) \tag{5}$$

Equating these two with (1) and (2), we obtain:

$$\cos\theta = -\cos(180° - \theta) \tag{6}$$
$$\sin\theta = \sin(180° - \theta) \tag{7}$$

For $\tan\theta$, we can use the identity given in the previous article. We had:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{8}$$

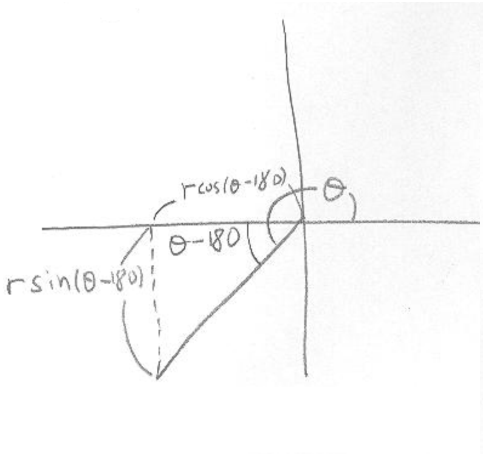

Figure 5: $180^\circ < \theta < 270^\circ$

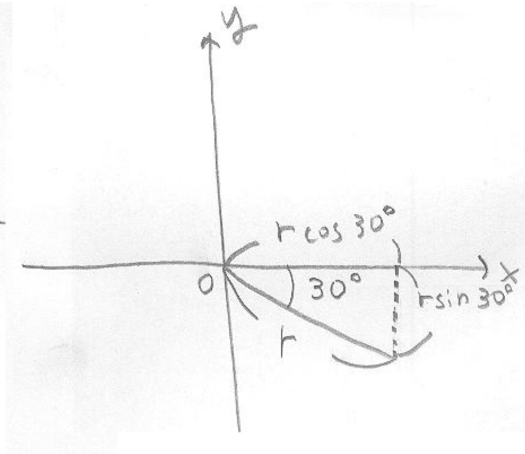

Figure 6: $\theta < 0$

Applying this, we get

$$\tan\theta = \frac{\sin(180^\circ - \theta)}{-\cos(180^\circ - \theta)} = -\tan(180^\circ - \theta) \tag{9}$$

Likewise, we can easily obtain the trigonometric functions for other ranges of angle easily.

For example, see Fig. 5. We see

$$x = r\cos\theta = -r\cos(\theta - 180^\circ) \tag{10}$$
$$y = r\sin\theta = -r\sin(\theta - 180^\circ) \tag{11}$$

So we conclude

$$\cos\theta = -\cos(\theta - 180^\circ) \tag{12}$$
$$\sin\theta = -\sin(\theta - 180^\circ) \tag{13}$$

Similarly, for $270^\circ < \theta < 360^\circ$, we can obtain the following

$$\cos\theta = \cos(360^\circ - \theta) \tag{14}$$
$$\sin\theta = -\sin(360^\circ - \theta) \tag{15}$$

The proof is left as an exercise to the leader.

Problem 1. By using (1), (2), and (8), obtain the following values.

$$\sin 0 = ?, \quad \cos 0 = ?, \quad \tan 0 = ? \quad \sin 90^\circ = ?, \quad \cos 90^\circ = ?$$
$$\sin 180^\circ = ? \quad \cos 180^\circ = ? \quad \tan 180^\circ = ?, \quad \sin 270^\circ = ? \quad \cos 270^\circ = ?$$

Problem 2. As an alternative to (4) and (5), show that the followings are also true.

$$\cos\theta = -\sin(90^\circ - \theta) \tag{16}$$
$$\sin\theta = \cos(90^\circ - \theta) \tag{17}$$

So, we can find the trigonometric functions for the range between 0° and 360°. However, could we define the trigonometric functions for the negative value of angle? The answer is yes. We can treat the negative value of angle as being rotated in the opposite direction from the x-axis. By the opposite direction I mean clockwise. For example, see Fig. 6.

We get the following:

$$x = r\cos(-30^\circ) = r\cos 30^\circ \tag{18}$$
$$y = r\sin(-30^\circ) = -r\sin 30^\circ \tag{19}$$

Therefore, we get

$$\cos(-30^\circ) = \cos 30^\circ \tag{20}$$
$$\sin(-30^\circ) = -\sin 30^\circ \tag{21}$$

By generalizing this argument, we get

$$\cos(-\theta) = \cos\theta \tag{22}$$
$$\sin(-\theta) = -\sin\theta \tag{23}$$

Problem 3. From (22) and (23), show the following:

$$\tan(-\theta) = -\tan\theta \tag{24}$$

We knew how the trigonometric functions are defined for the range between 0° and 360°. However by above formula we know how they are defined for the range between -360° and 0°. A natural question one may ask is: "Could we define the value of the trigonometric functions for all range?" The answer is yes. Think of cos 400°, and sin 400°. See Fig. 7.

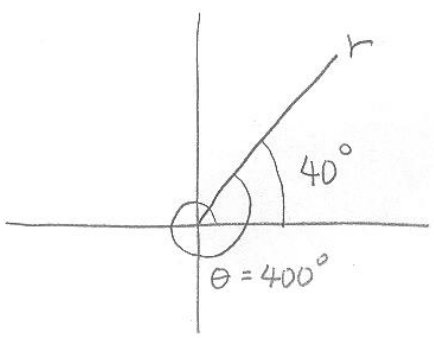

Figure 7: $\theta = 400^\circ$

If you rotate 360° anticlockwise from x axis, you come to the original x axis again. So, if you rotate 400° anticlockwise from x axis, you are on the same point as the point you reach after rotating 40° from x axis as anticlockwise. Therefore, we can easily see that

$$x = r\cos 400^\circ = r\cos 40^\circ \tag{25}$$
$$y = r\sin 400^\circ = r\sin 40^\circ \tag{26}$$

Thus, we have

$$\cos 400^\circ = \cos 40^\circ \tag{27}$$
$$\sin 400^\circ = \sin 40^\circ \tag{28}$$

Let's generalize what we just have done. If you rotate a point x angle anticlockwise it arrives at the same position as the point which you get by rotating 360° once more or once less. Thus,

$$r\cos x = r\cos(x + 360^\circ) = r\cos(x - 360^\circ) \tag{29}$$
$$r\sin x = r\sin(x + 360^\circ) = r\sin(x - 360^\circ) \tag{30}$$

which implies

$$\cos x = \cos(x + 360^\circ) = \cos(x - 360^\circ) \tag{31}$$
$$\sin x = \sin(x + 360^\circ) = \sin(x - 360^\circ) \tag{32}$$

Now, we can define trigonometric functions for all range. For example,

$$\begin{aligned}\sin 800^\circ &= \sin(800^\circ - 360^\circ) = \sin 440^\circ \\ &= \sin(440^\circ - 360^\circ) = \sin 80^\circ\end{aligned} \tag{33}$$

For another example,

$$\begin{aligned}\cos(-780^\circ) &= \cos(-780^\circ + 360^\circ) = \cos(-420^\circ) \\ &= \cos(-420^\circ + 360^\circ) = \cos(-60^\circ) = \cos 60^\circ = 1/2\end{aligned} \tag{34}$$

We easily see that trigonometric functions are periodic; a function is called "periodic" if it satisfies the following condition:

$$f(x + T) = f(x) \tag{35}$$

Here, T is called the "period." In our case, the periods of sine function and cosine function are both 360°.

Problem 4. Show that a periodic function of period T satisfies following. (Answer[1])

$$f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = f(x + 4T) = \cdots \tag{36}$$

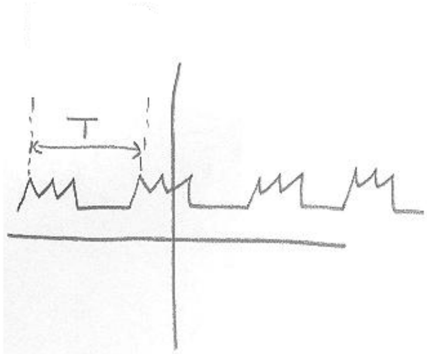

Figure 8: a periodic fucntion with period T

It implies that the function gets repeated as x goes on. See Fig. 8 for an example of periodic functions. We see here that the period is given by T. A good example of periodic function is a wave. You have highs and lows periodically. Being periodic, trigonometric functions are used to describe waves in physics. In fact, it is known that every periodic function can be represented as sums of sine and cosine functions. Interested reader may want to read our later article "What is Fourier series?"

Problem 5. Which of the following is the period of the function $\sin(x/2)$?
(1) $180°$ (2) $360°$ (3) $720°$

Problem 6. Without using an calculator, obtain the values for following:

$$\sin 120 = ?, \quad \tan 135° = ?, \quad \sin 150° = ?, \quad \tan 240° = ??$$
$$\sin 360° = ?, \quad \cos 360° = ?, \quad \cos(-30°) = ?, \quad \tan(-45°) = ?$$

Problem 7. Express the following Cartesian coordinate (i.e. (x, y)) in terms of polar coordinate:

$$(4, 0), \quad (0, 5), \quad (3, -3), \quad (-1, 0)$$

Problem 8. There is no reason that r should be positive or zero. Show that the polar coordinates (r, θ) and $(-r, \theta + 180°)$ designate the same point.

Problem 9. In Fig. 9 we have two graphs: one with dotted line, and one with solid line. One of them is the sine function and another the cosine function. Which one is which? (Hint[2])

Problem 10. Given this, in Fig. 9, what is the x-coordinate of the point the arrow is pointing? Notice that the y-coordinate of this point is 0.

[1](35) can be re-written as $f(y+T) = f(y)$. Plug in $y = x+T$, $y = x+2T$, $y = x+3T$.
[2]Use the results of Problem 1.

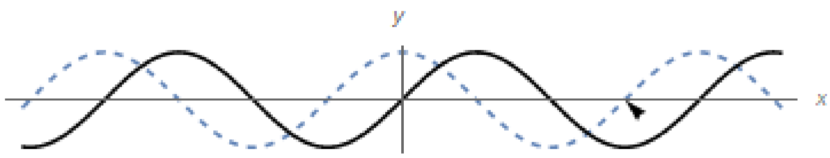

Figure 9: sine and cosine functions

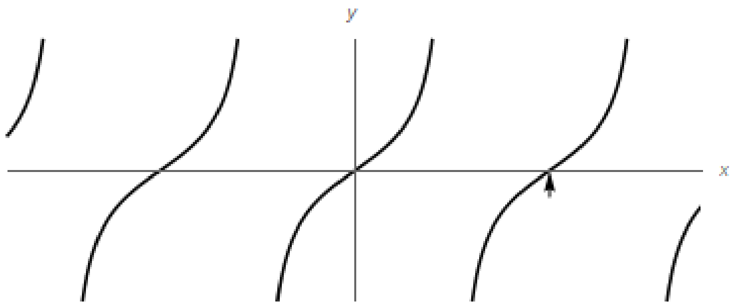

Figure 10: tangent function (Problem 11)

Problem 11. In Fig. 10 we have a graph for tangent function. What is the x-coordinate of the point the arrow is pointing? Notice that the y-coordinate of this point is 0. Thus, obtain the period of the tangent function.

Problem 12. Alternatively, obtain the period of the tangent function from (12) and (13).

Problem 13. Show the following! (Hint[3])

$$\tan\theta \tan(\theta + 90^\circ) = -1 \tag{37}$$

Alternatively, you can think of it this way. A slope of the graph is defined by

$$m = \frac{\Delta y}{\Delta x} \tag{38}$$

If the angle that a graph makes with x-axis is θ (more precisely speaking, if the graph is obtained by θ degrees anti-clockwise rotation from the x-axis, see the figure in "Slopes of two perpendicular lines") then we have

$$m = \tan\theta \tag{39}$$

Now, a graph that is perpendicular to this graph can be obtained by $\theta + 90^\circ$ anti-clock rotaion from the x-axis. Thus,

$$m' = \tan\theta' = \tan(\theta + 90^\circ) \tag{40}$$

As we have $mm' = -1$, from (39) and (40), we obtain (37).

[3]Use $\tan(-\alpha) = -\tan\alpha$ and $\tan(90^\circ - \beta) = 1/\tan\beta$.

Summary

- Polar coordinate is defined by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

  Thus, we naturally have

$$r = \sqrt{x^2 + y^2}, \qquad \tan\theta = \frac{y}{x}$$

- The trigonometric functions can be defined beyond the range $0 < \theta < 90°$.
- $\sin 0 = 0, \qquad \sin 90° = 1$
- $\cos 0 = 1, \qquad \cos 90° = 0.$
- $\tan 0 = 0.$
- $\sin(-\theta) = -\sin\theta, \qquad \cos(-\theta) = \cos\theta$
- Thus, we have

$$\tan(-\theta) = -\tan\theta$$

  .
- $\sin(\theta + 180°) = -\sin\theta, \qquad \cos(\theta + 180°) = -\cos\theta$
- Thus, we have

$$\tan(\theta + 180°) = \tan\theta$$

  .
- $\sin(\theta + 360°) = \sin\theta, \qquad \cos(\theta + 360°) = \cos\theta.$