Polar coordinate, the area of a circle and Gaussian integral

In this article, we will deal with multiple integrations in polar coordinate; we will explicitly calculate the area of a circle rigorously and obtain the value for Gaussian integral encountered in our earlier article "Expectation values in quantum field theory (1)."

To this end, let's first talk about "area element." To calculate the area of something, you need to integrate area element. For example, see Fig.1. The area of shaded area is given by

$$\int_a^b \int_{g(x)}^{f(x)} dydx \tag{1}$$

So $dydx$ is integrated. This is the area element, often denoted as dA. This is intuitively clear if you look at Fig. 2. The small rectangle has the area $dA = dydx$. This is the area element in Cartesian coordinate.

Then, what would be the area element for polar coordinate? See Fig.3. The small bended rectangle has the area $rdrd\theta$. This is the area element. Actually, we can obtain this in another way using Jacobian. Remember,

$$x = r\cos\theta \tag{2}$$

$$y = r\sin\theta \tag{3}$$

and our change of variable equation:

$$\int\int dxdy = \int\int \frac{\partial(x,y)}{\partial(r,\theta)} drd\theta \tag{4}$$

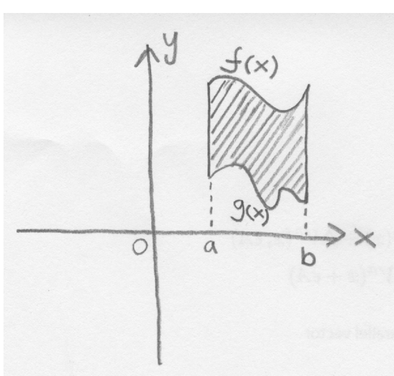

Figure 1: calculating area

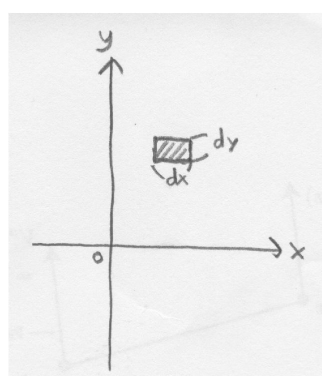

Figure 2: area element

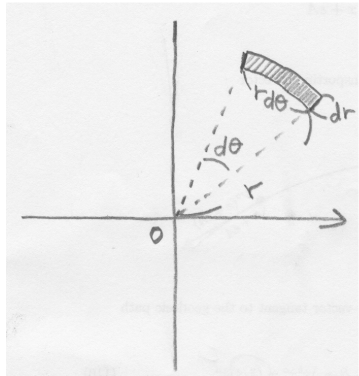

Figure 3: calculating area

If you calculate the Jacobian, you get r as expected. (Problem 1. Check this)

Now, let's calculate the area of circle with radius a. We can calculate it by integrating the area element as follows:

$$\int_{r=0}^{r=a} \left(\int_{\theta=0}^{\theta=2\pi} r d\theta \right) dr = \int_0^a 2\pi r dr = \pi a^2 \tag{5}$$

Therefore, we obtained an expression which you are already familiar with.

Now, our next target is Gaussian integral. Gaussian integral, which we call I here, is given by:

$$I = \int_{-\infty}^{+\infty} e^{-x^2} dx \tag{6}$$

Now, notice following:

$$I = \int_{-\infty}^{+\infty} e^{-y^2} dy \tag{7}$$

$$\begin{aligned} I^2 &= \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy & (8) \\ &= \int_{r=0}^{r=\infty} \int_{\theta=0}^{2\pi} e^{-r^2} r dr d\theta = \int_{r=0}^{r=\infty} 2\pi r e^{-r^2} & (9) \end{aligned}$$

Further noticing the following:

$$\frac{d(-\pi e^{-r^2})}{dr} = 2\pi r e^{-r^2} \tag{10}$$

we obtain:

$$I^2 = (-\pi e^{-\infty^2}) - (-\pi e^{-0^2}) = \pi \tag{11}$$

Therefore,

$$I = \sqrt{\pi} \tag{12}$$

This is the result of Gaussian integral.

Summary

- We can obtain the value for the Gaussian integral by calculating the integration $\int \exp(-x^2 - y^2)xdy$ in polar coordinate and taking its square root.