Derivatives of the polynomials

Let $h(x) = 4.3$. What is its derivative with respect to x? Certainly,

$$\frac{dh}{dx} = \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x} = \frac{4.3 - 4.3}{\Delta x} = 0 \tag{1}$$

Therefore, if we take derivatives of any constant function, it is zero.

Another example: Let $f(x) = x$. What is its derivative with respect to x?

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{x + \Delta x - x}{\Delta x} = 1 \tag{2}$$

In other words, the slope of $y = x$ is 1.

How about x^2?

$$\frac{d(x^2)}{dx} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = 2x + \Delta x = 2x \tag{3}$$

How about x^3? If you remember our earlier article on Pascal's triangle, you will see the following:

$$\frac{d(x^3)}{dx} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \frac{3x^2 \Delta x + 3x \Delta x^2 + \Delta x^3}{\Delta x} = 3x^2 + 3x\Delta x + \Delta x^2 = 3x^2 \tag{4}$$

Similarly, using the binomial expansion as explained in "The imagination in mathematics:"Pascal's triangle, combination, and the Taylor series for square root," one can derive:

$$\frac{d(x^n)}{dx} = nx^{n-1}, \quad \text{or} \quad (x^n)' = nx^{n-1}, \tag{5}$$

Actually, one can alternatively derive this using the method of induction. Let's do this. First, (5) is obviously satisfied for $n = 1$. When $n = 1$, we indeed have

$$\frac{dx}{dx} = 1x^0 = 1 \tag{6}$$

Second, we need to check that (5) is satisfied for $n = m + 1$, if it is satisfied for $n = m$. Let's do this. When $n = m$, we have

$$\frac{dx^m}{dx} = mx^{m-1} \tag{7}$$

When $n = m + 1$, using Leibniz's rule, we have

$$\frac{d(x^m x)}{dx} = \frac{dx^m}{dx} x + x^m \frac{dx}{dx} = mx^{m-1}x + x^m = mx^m + x^m = (m+1)x^m \tag{8}$$

which indeed satisfies (5) for $n = m + 1$. This completes the proof.

In fact, our proof is only valid for a positive integer n, but (5) holds not only when n is a positive integer, but also when n is any real number. (We will prove this in our later article "Differential and infinitesimal change.") For example:

$$\frac{d(\sqrt{x})}{dx} = \frac{d(x^{1/2})}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \tag{9}$$

$$\frac{d(x^{-2})}{dx} = -2x^{-3} \tag{10}$$

and so on.

Then, how can we find the derivatives of $3x^2$? It is 3 times the derivatives of x^2. Remember that the derivative of $cf(x)$ is given by c times the derivative of $f(x)$. Thus,

$$(3x^2)' = 3(x^2)' = 3 \cdot 2x = 6x \tag{11}$$

Now, we can easily find the derivatives of any polynomial functions. For example:

$$(5x^2 + 3x^6 + 4)' = 5 \cdot 2x + 3 \cdot 6x^5 = 10x + 18x^5 \tag{12}$$

One can also differentiate the same function multiple times. For example, if we differentiate a function f two times, we express this as:

$$\frac{d^2 f}{dx^2} \equiv \frac{d}{dx}\left(\frac{df}{dx}\right), \qquad \text{or} \quad f'' \tag{13}$$

We pronounce f'' as "f two prime." For an explicit example, if $f(x) = 10x^2$,

$$(10x^2)'' = (20x)' = 20 \tag{14}$$

As another example, using Leibniz's rule, we have

$$\begin{aligned}(xf(x))'' &= (f(x) + xf'(x))' = f'(x) + (xf'(x))' \\ &= f'(x) + f'(x) + xf''(x) = 2f'(x) + xf''(x)\end{aligned} \tag{15}$$

Problem 1.

$$\left(-\frac{1}{\sqrt{x}} - 3x^2 + 4x\right)' = ?, \qquad \frac{d^2(x^3 + 3x + 1)}{dx^2} = ? \tag{16}$$

Problem 2. (Hint[1])

$$\frac{d\sqrt{2x}}{dx} = ?, \qquad \frac{d^2(x^m)}{dx^2} = ? \tag{17}$$

Problem 3. Express $(x^2 f(x))''$ in terms of x, $f'(x)$ and $f''(x)$. (Hint[2])

Summary

- If we differentiate any constant function, we get zero.
- $\dfrac{d(x^n)}{dx} = nx^{n-1}$

[1] Use $\sqrt{2x} = \sqrt{2}\sqrt{x}$.

[2] Use Leibniz's rule.