Potential energy and conservation of energy

As before, in this article, we will assume that the friction due to air is negligible. Suppose you throw a ball with mass m downward with speed v_i at the initial height h_i. See Fig. 1. What is the speed v_f at the same instance when it reaches the height h_f? We can easily solve this problem by using the formula introduced in our earlier article "Constant acceleration in 1-dimension" as follows.

$$s = \frac{v_f^2 - v_i^2}{2a} \tag{1}$$

By plugging $s = h_2 - h_1$, $a = g$ we get:

$$\begin{aligned} h_f - h_i &= \frac{v_f^2 - v_i^2}{2g} \\ v_f^2 - v_i^2 &= 2g(h_f - h_i) \\ v_f &= \sqrt{v_i^2 + sg(h_f - h_i)} \end{aligned} \tag{2}$$

However, we can obtain the same formula using the concept of work, which turns out to be more powerful, as we will see soon. What is the work done on the object when it moves from the height h_i to the height h_f? Remembering that the gravitational force mg is exerted on the object downward, and noticing the fact the distance moved is given by $h_i - h_f$, we easily see that the work done is $mg(h_i - h_f)$. As we also know that the change of kinetic energy is given by the work done, we conclude:

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = mg(h_f - h_i) \tag{3}$$

which yields (2).

Now, consider a slightly different situation. See Fig.2. Suppose a ball moves downward making an angle θ with the vertical direction. The height of the initial position is h_1 and the final position is h_2. We will also assume that h_1 and h_2 are close to each other so, θ doesn't change much during the flight. Then, what is the work done by the gravitational force on the ball? We have:

$$W = mgs\cos\theta = mg(s\cos\theta) = mg(h_1 - h_2) \tag{4}$$

Therefore, we see that it is independent of θ.

Now, let's lift the condition that θ doesn't change much. See Fig.3. Suppose the ball was at height h_i initially and h_f finally. What is the total work done by gravitational force on the ball during this flight? To calculate the work done, let's chop up the flight by small

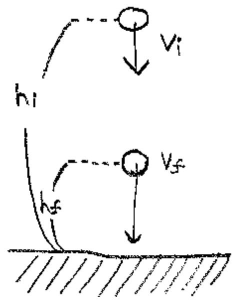

Figure 1:

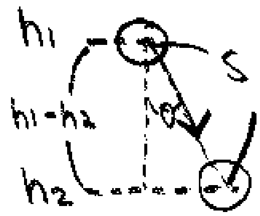

Figure 2:

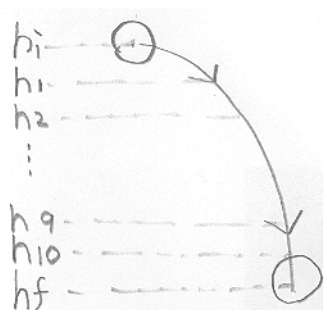

Figure 3:

intervals in which our θ, the direction in which the ball moves. Here, we have chopped up 10 times, inserting the heights $h_1, h_2 \cdots h_{10}$ between h_i and h_f. From (4), the gravitational work done on the first interval $h_i < h < h_1$ is given by $mg(h_i - h_1)$. The second interval $h_1 < h < h_2$ is given by $mg(h_1 - h_2)$ and so forth. The total is then given as follows:

$$\begin{aligned} W &= mg(h_i - h_1) + mg(h_1 - h_2) + mg(h_2 - h_3) + \cdots mg(h_9 - h_{10}) + mg(h_{10} - h_f) \\ W &= mg(h_i - h_f) \end{aligned} \tag{5}$$

The final result seems simple. So, the work done doesn't depend on the direction the ball is moving but only on the height difference. If the initial velocity of the ball is v_i and the final one v_f, we have:

$$\begin{aligned} \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 &= mg(h_i - h_f) \\ \frac{1}{2}mv_f^2 + mgh_f &= \frac{1}{2}mv_i^2 + mgh_i \end{aligned} \tag{6}$$

Given this, let's define the gravitational potential energy as follows:

$$E_p = mgh \tag{7}$$

where h is the height. Then, we can re-express (6) as follows:

$$E_{kf} + E_{pf} = E_{ki} + E_{pi} \tag{8}$$

We see that the sum of the kinetic energy and the potential energy doesn't change; the total energy is constant. This is called the "conservation of energy." We derived this using the assumption that $h_i > h_f$, but in fact it doesn't really matter whether this condition is satisfied. All the similar steps can be taken to arrive at the same conclusion when $h_i < h_f$.

Given this result, let's apply it to a problem. Suppose you throw a ball with the speed 10m/s at the height $h_i = 0\text{m}$. What would be its speed when it reaches the height $h_f = 5\text{m}$? If the ball's mass is m, the initial total energy is given as follows:

$$\frac{1}{2}m \cdot 10^2 + mg \cdot 0 = 50m \tag{9}$$

If the ball's speed at $z = 5\text{m}$ is v_f, the total energy at that point is given by:

$$\frac{1}{2}mv^2 + mg \cdot 5 = \frac{1}{2}mv_f^2 + 49m \tag{10}$$

Since this value must be $50m$ by the conservation of energy, we conclude:

$$\begin{aligned} \frac{1}{2}mv^2 &= m \\ v^2 &= 2 \\ v = \sqrt{2}\text{m/s} \approx 1.4\text{m/s} \end{aligned} \tag{11}$$

This is the answer. Notice that this value doesn't depend on the initial ball direction, as long as it reaches the height 5m. Remember that the kinetic energy of the ball depends only on the speed, not on the direction. Nor does it depend on the mass of the ball.

Let me conclude this article with a musing on the meaning of potential energy. The word "potential" implies that potential energy has a potential to be converted into kinetic energy which is more manifest. Also, from (7), we see that the higher an object the more potential energy it has. It makes perfect sense as higher object gets more kinetic energy when it falls down to the ground compared to the case in which an object with the same mass with a lower initial height falls down to the ground. Notice also that potential energy is converted to kinetic energy when the object is moving downwards, while kinetic energy is converted to potential energy when the object is moving upwards. It is also worthwhile to mention that there are other types of potential energy than the gravitational one. For example, unused battery has more potential energy than used battery, and so forth.

Problem 1. If a ball is projected from the ground with a velocity of 20 m/s at an angle 40° above the horizontal. What will be its speed when it hits the ground again? Ignore air friction.

Summary

- The sum of kinetic energy and potential energy is conserved.
- The gravitational potential energy is given by mgh, where m is the mass, g is the gravitational acceleration, and h is the height.