The power of negative number

In elementary school, you learn that you cannot subtract 5 from 2, because 5 is bigger than 2. Then, in middle school, you are introduced to the concept of "negative number," and you can suddenly subtract 5 from 2. It is "-3." You encounter a number you never heard of. Do negative numbers really "exist?"

Maybe, your teacher introduced the negative numbers as follows. If you have only 2 dollars, but have to pay 5 dollars. So, you get 3 dollar loan and pay the bill. Now, you are 3 dollar in debt. In other words, you have -3 dollars. From this point of view, the concept of negative number seems to be not really necessary, but only convenient. It seems that you could get along without using negative numbers, even though it may be a little bit incovenient.

To explain what is wrong with this point of view, let us mention the following story.

A cubic equation is a type of equation that can be represented in the form $ax^3 + bx^2 + cx + d = 0$, where a is non-zero. (If $a = 0$, it is a quadratic equation, which is in the form of $bx^2 + cx + d = 0$.)[1]

In the early 16th century, Scipione del Ferro obtained a general solution to the cubic equation of the form $x^3 = sx + t$ where s and t are positive numbers. At the time, mathematicians didn't have a firm concept of negative numbers. Therefore, del Ferro's solution couldn't be applied to solve the cubic equation of the form $x^3 + px = q$, where p and q are positive numbers; these two cubic equations were regarded as two different types. Now, any smart high school student can solve the cubic equation of the form $x^3 + px = q$, such as $x^3 + 2x = 3$ if she knows the general solution to $x^3 = sx + t$.

Let's see how she can do. $x^3 + 2x = 3$ is equivalent to $x^3 = -2x + 3$. All she needs to do is plug in $s = -p$, i.e., $s = -2$ and $t = q$, i.e., $t = 3$ to the general solution of $x^3 = sx+t$, as del Ferro's solution should work no matter whether s and t are positive or negative. However, the substitution $s = -2$ was impossible in the 16th century, because it required the concept of negative number. Even though mathematicians began to have ideas on negative numbers around the 15th century, the concept of negative numbers was firmly established only in the 17th century.

Something a high school student now can easily take for granted was neither taken for granted nor easily thought of by the first rate mathematicians 500 years ago. The concept of negative number seems very easy now, but it first seemed very foreign and unfamiliar even to the smartest people. Think about the fact that it took two centuries to accept the negative numbers!

A comment. Now, you know that there are positive numbers, zero, and negative numbers in the categories of numbers. Are there any other numbers? Can we further extend the boundary of numbers, as we once did by including negative numbers? The answer is yes. Mathematicians extended the boundary of numbers by including what they call now "imaginary numbers." Imaginary numbers are not zero, but they are

[1]We assume here that b is not zero. If both a and b are zero, it is not a quadratic equation, but a linear equation.

neither bigger than zero nor smaller than zero. In other words, they are neither positive numbers nor negative numbers. Do imaginary numbers really exist? Yes. In our later article "Complex number," I will explain why the concept of imaginary number is more than just a convenient tool, as I explained why the concept of negative numbers is more than just a convenient tool in this essay. Once again, the cubic equation proves this point.

For those of you, who know how to solve quadratic equations, please accept my invitation to solve the following cubic equation problems. Otherwise, you can try these problems after you learn how to solve quadratic equations.

Problem 1. In this problem, we will do the other way around. In 1546, the Italian mathematician Tartaglia found

> If $x^3 + px = q$, find u, v such that
>
> $$u - v = q, \qquad uv = \left(\frac{p}{3}\right)^3 \tag{1}$$
>
> Then, $x = \sqrt[3]{u} - \sqrt[3]{v}$.

Given this, how can we find the solution to $x^3 = sx + t$, where s and t are positive? Let's say $x = \sqrt[3]{a} + \sqrt[3]{b}$. In (1), u and v satisfied the two equations $u - v = q$ and $uv = (p/3)^3$. Similarly, what two equations do a and b satisfy? Express them in terms of s and t, as (1) is expressed in terms of p and q.

Problem 2. Obtain u and v for (1) in terms of p and q. (Hint[2])

Problem 3. What is the solution to $x^3 = 3x - 2$? (There are two solutions to this cubic equation, but for the purpose of this problem, it is sufficient to obtain only one.)

[2] We have $u = v + q$, which implies $(v + q)v = (p/3)^3$. This is a quadratic equation for v.