Fundamental theorem of algebra

In our earlier article "Fundamental theorem of algebra" we stated that any non-constant polynomial $f(x)$ has at least one solution to $f(x) = 0$, and therefore, a degree n polyonomial $f(x)$ is completely factorizable as $f(x) = c(x - a_1)(x - a_2) \cdots (x - a_n)$. In this article, we will prove the fundamental theorem of algebra.

To prove this we first need to introduce the concept of "entire" function. A function is called entire if it is holomorphic at every point on complex plane. For example, $f(z) = z + 2z^2 + z^4$ is entire, while $g(z) = 2/(z - 1)$ is not as it is not holomorphic at $z = 1$.

We also need to introduce the concept of "bounded." A bounded function has a maximum limit. For example, if the maximum is M, a bounded function $h(z)$ satisfies $|h(z)| \leq M$ for any z on entire complex plane.

Now comes an interesting result. Liouville's theorem states that a bounded entire function is always a constant function. This sounds reasonable, considering that a function cannot be bounded if you have terms like z, $2z^2$, z^3 in its Taylor expansion, as they become very big as z approaches infinity. Then, the only possibility for an entire function to be bounded is that there is only one term, a constant term in the Taylor expansion of the function. But, this is just a gut feeling. How can we prove Liouville's theorem rigorously?

Suppose $f(z)$ is a bounded entire function. Then, Cauchy's integral formula says

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz, \qquad f(b) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - b} dz, \tag{1}$$

where C encircles both $z = a$ and $z = b$. This is true, because the function $f(z)/(z - a)$ has no other poles than $z = a$ as $f(z)$ is an entire function. Same can be said about $f(z)/(z - b)$. Thus, we have

$$f(a) - f(b) = \frac{1}{2\pi i} \oint_C \left(\frac{f(z)}{z - a} - \frac{f(z)}{z - b} \right) dz = \frac{1}{2\pi i} \oint_C \frac{f(z)(a - b)}{(z - a)(z - b)} dz \tag{2}$$

Now, let's choose the contour C to be a circle of radius R with its center located at $z = 0$. Just like we showed in our last article that the contribution from the semi-circle can be made arbitrarily small by taking a large radius, we will now show that the above integral can be made arbitrarily small if we take the radius of circle arbitrarily big. As $f(z)$ is a bounded function, we can write $|f(z)| \leq M$ for some M. Then, the above expression can be made arbitrarily small as

$$|f(a) - f(b)| = \left| \frac{1}{2\pi i} \oint_C \frac{f(z)(a - b)}{(z - a)(z - b)} dz \right| \leq \frac{M|a - b|R}{|R - a||R - b|} \tag{3}$$

If we send R to infinty, the right side of the inequality goes to zero. Thus, we conclude $f(a) = f(b)$. In other words, $f(z)$ is a constant function.

Now, we can prove the fundamental theorm of algebra. Let $g(z)$ be a polynomial. If we assume that it has no solution for $g(z) = 0$, then, $1/g(z)$ is an entire function, because it has no poles, which would have made it non-differentiable at the poles. Also, as $|g(z)|$ has a minimum limit, $|1/g(z)|$ has a maximum limit. Thus, $1/g(z)$ is a bounded entire function, which means that it is a constant function. In conclusion, $g(z)$ must be a constant polynomial, if $g(z) = 0$ does not have any solution. In other words, if $g(z)$ is a non-constant polynomial, $g(z) = 0$ has at least one solution.

Summary

- A function is called entire if it is holomorphic at every point on complex plane.
- A bounded function has a maximum limit. If the maximum is M, a bounded function $h(z)$ satisfies $|h(z)| \leq M$ for any z on entire complex plane.
- A bounded entire function is always a constant function.
- If a polynomial $f(z)$ has no zeros, $1/f(z)$ is a bounded entire function, which means that it is a constant function. Therefore, unless a polynomial is a constant function, it always has at least one solution for $f(z) = 0$.