Propagation of error

Suppose the length of stick A is (11.00 ± 0.05) cm and the length of stick B is (12.00 ± 0.05) cm. If you connect these sticks one after another, what will be the total length of the sticks?

If you recall our earlier article "Scientific notation," you will think that the length of stick A is between 10.95 cm and 11.05 cm and the length of stick B is 11.95 cm and 12.05 cm, so the total length is between 22.90(=10.95+11.95) cm and 23.10(=11.05+12.05) cm. So, you may conclude (23.0 ± 0.1) cm.

Your reasoning is certainly right. However, according to the common notation in science, you are wrong. The problem is that I lied in the article "Scientific notation," strictly speaking. When a scientist says that the length of stick A is (11.00 ± 0.05) cm, he or she *doesn't* mean that it is between 10.95 cm and 11.05 cm. By the uncertainty 0.05 cm in (11.00 ± 0.05) cm, scientists mean the standard deviation. If the length of A were indeed between 10.95 cm and 11.05 cm, the deviation from the mean, 11.00 cm, is always less than or equal to 0.05 cm. So, in such a case, the standard deviation, which is some kind of average of the deviation from the mean, is necessarily slightly is smaller than 0.05 cm.

Putting this back slightly differently, if the length of A is (11.00 ± 0.05) cm, there is necessarily chance that it is outside of the range 10.95 cm and 11.05 cm, even though such chance may not that big.

Then, what is the correct total length? If you read "Standard deviation of the sum of uncorrelated data," you can see that the variance adds up, if there is no correlation between the first data and the second data. So, assuming that there is no correlation between the length of A and the length of B,[1] we have

$$\mathrm{Var}(\mathrm{A} + \mathrm{B}) = \mathrm{Var}(\mathrm{A}) + \mathrm{Var}(\mathrm{B}) = 0.05^2 + 0.05^2 = 0.005 \tag{1}$$

Thus, the standard deviation is 0.07 ($\approx \sqrt{0.005}$) cm. Thus, the length of A+B is (23.00 ± 0.07) cm.

So, you see that the uncertainty is slightly smaller than 0.1 cm, our earlier naive value. Why is this so? Because it is not that likely that both the length of A and the length of B deviate from the mean length with big values. For example, if A is 11.05 cm and B is 12.05 cm, we have the total deviation of 0.1 cm. However, it is not that likely that they both have

[1] Of course, in a real situation, there can be a correlation. For example, if the same company manufactured stick A and stick B, and if the company tends to make sticks shorter than the target length, there will be a correlation between A and B.

that big values *at the same time*. If you read our earlier article "Standard deviation of the sample means," you will recall that there is only 1/36 chance that you get two 6s by throwing dices. Moreover, notice that there is the equal chance that A is 11.05 cm and B is 11.95 cm as A is 11.05 cm and B is 12.05 cm. In the former case, we get A+B is 23 cm, which has the deviation of 0 cm. Thus, you see that such factor drives down the standard deviation, well below 0.1 cm.

More generally, if we have $f(a, b, c)$, a function of a, b, and c where a, b and c are independent variables, with the standard deviation (uncertainty) of Δa, Δb, and Δc, respectively, we can obtain the standard deviation of f by the following procedure. First, note

$$df = \frac{df}{da}da + \frac{df}{db}db + \frac{df}{dc}dc \tag{2}$$

Then, the variance of f is given by

$$(\Delta f)^2 = \left(\frac{df}{da}\Delta a\right)^2 + \left(\frac{df}{db}\Delta b\right)^2 + \left(\frac{df}{dc}\Delta c\right)^2 \tag{3}$$

By taking the square root, you can get Δf, the standard deviation (uncertainty, error) of f.

Problem 1. If $s = (4.90 \pm 0.01)$ m, $t = (1.00 \pm 0.01)$ sec, and

$$s = \frac{1}{2}gt^2 \tag{4}$$

What is g?

Let me give you other examples, even though they are redundant. Suppose $f = xyz$. If x has 1% uncertainty, y, 0.5 % uncertainty, and z, 0.8 % uncertainty, what is the relative uncertainty of f? We could have used (3), which is the reason why I called it a redundant example, but there is a quicker way. First, take log. Then,

$$\ln f = \ln x + \ln y + \ln z \tag{5}$$

By differentiating, we get

$$\frac{df}{f} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} \tag{6}$$

Thus,

$$\left(\frac{\Delta f}{f}\right)^2 = \left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2 + \left(\frac{\Delta z}{z}\right)^2 \tag{7}$$

$$= (1\%)^2 + (0.5\%)^2 + (0.8\%)^2 = 1.89(\%)^2 \tag{8}$$

$$\frac{\Delta f}{f} = \sqrt{1.89}\% \approx 1.4\% \tag{9}$$

Problem 2. Suppose $f = x/y$. If x has 1% uncertainty and y has 1% uncertainty, what is the relative uncertainty of f?

Problem 3. Suppose $f = 4x^2/y^3$. If x has 0.5% uncertainty and y has 0.5% uncertainty, what is the relative uncertainty of f?

Summary

- Let's say a has the uncertainty of Δa, b the uncertainty of Δb, and c the uncertainty of Δc. Then, the uncertainty of $f(a,b,c)$ can be obtained by the following procedure:

$$df = \frac{df}{da}da + \frac{df}{db}db + \frac{df}{dc}dc$$

$$(\Delta f)^2 = \left(\frac{df}{da}\Delta a\right)^2 + \left(\frac{df}{db}\Delta b\right)^2 + \left(\frac{df}{dc}\Delta c\right)^2$$

Thus,

$$\Delta f = \sqrt{\left(\frac{df}{da}\Delta a\right)^2 + \left(\frac{df}{db}\Delta b\right)^2 + \left(\frac{df}{dc}\Delta c\right)^2}$$