## Expectation values in quantum field theory (1)

To understand quantum field theory you must know quantum mechanics and special relativity. However, you can still learn the basic ideas about the way physicists calculate expectation values in quantum field theory without knowledge of physics.

The expectation value for $f(x)$ is defined by the following formula:

$$
\begin{equation*}
\langle f(x)\rangle=\frac{\int_{-\infty}^{\infty} f(x) e^{-A x^{2}} d x}{\int_{-\infty}^{\infty} e^{-A x^{2}} d x} \tag{1}
\end{equation*}
$$

Here, one can understand the expectation value as the expected value, or the average outcome of $f(x)$. For example, in a lottery, if one person wins $\$ 1,000,000,3$ people win $\$ 10,000,100$ people win $\$ 100$, and $1,000,000$ people win $\$ 0$, the expectation value for a lottery ticket is given by the following formula:
$\langle X\rangle=\frac{\$ 1000000 * 1+\$ 10000 * 3+\$ 100 * 100+\$ 0 * 1000000}{1+3+100+1000000}=\frac{\int X * N(X)}{\int n(X)}$
where $N(X)$ is the number of tickets that will be awarded $\$ X$. In the continuum limit, this formula reduces to our first formula; the summation is replaced by an integral.

Now, let's try to calculate this formula. To this end, we should use the following formula:

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{-x^{2}} d x=\sqrt{\pi} \tag{3}
\end{equation*}
$$

Deriving this formula is beyond the scope of this article, since it requires a solid knowledge of multivariable calculus; we will derive it in a later article "Polar coordinate, the area of a circle and Gaussian integral." At the moment, we can take this formula for granted, and use it to advance our discussion further.

Thus, we can obtain the following:

$$
\begin{array}{r}
\int_{-\infty}^{\infty} e^{-A x^{2}+B x} d x=\int_{-\infty}^{\infty} e^{-A\left(x-\frac{B}{2 A}\right)^{2}+\frac{B^{2}}{4 A}} d x \\
=\frac{e^{\frac{B^{2}}{4 A}}}{\sqrt{A}} \int_{-\infty}^{\infty} e^{-y^{2}} d y=\sqrt{\frac{\pi}{A}} e^{\frac{B^{2}}{4 A}} \tag{5}
\end{array}
$$

where we have used the following substitution:

$$
\begin{equation*}
y=\sqrt{A}\left(x-\frac{B}{2 A}\right) \tag{6}
\end{equation*}
$$

Now, let's assume that $f(x)$ is a polynomial. This is justified because if $f(x)$ is not a polynomial, we can still approximate it with one by taking the Taylor-expansion of $f(x)$, and ignoring higher order terms whose effects are negligible. With this assumption, $f(x)$ is given by the following formula:

$$
\begin{equation*}
f(x)=f_{0}+f_{1} x+f_{2} x^{2}+f_{3} x^{3}+\ldots \tag{7}
\end{equation*}
$$

Hence it is sufficient to compute

$$
\begin{equation*}
\int_{-\infty}^{\infty} x^{p} e^{-A x^{2}} d x \tag{8}
\end{equation*}
$$

where $p$ is a non-negative integer.
Let's first calculate this when $p=1$.
Notice that

$$
\begin{equation*}
\frac{\partial \int_{-\infty}^{\infty} e^{-A x^{2}+B x} d x}{\partial B}=\int_{-\infty}^{\infty} x e^{-A x^{2}+B x} d x \tag{9}
\end{equation*}
$$

Plugging in our previous result, we obtain:

$$
\begin{equation*}
\int_{-\infty}^{\infty} x e^{-A x^{2}+B x} d x=\sqrt{\frac{\pi}{A}} \frac{\partial e^{\frac{B^{2}}{4 A}}}{\partial B}=\sqrt{\frac{\pi}{A}} \frac{B}{2 A} e^{\frac{B^{2}}{4 A}} \tag{10}
\end{equation*}
$$

Therefore, by plugging in $B=0$, we obtain 0 .
Now let's computer the case $p=2$. We have:

$$
\begin{align*}
& \frac{\partial^{2} \int_{-\infty}^{\infty} e^{-A x^{2}+B x} d x}{\partial B^{2}}=\int_{-\infty}^{\infty} x^{2} e^{-A x^{2}+B x} d x \\
& =\sqrt{\frac{\pi}{A}} \frac{1}{2 A} \frac{\partial\left(B e^{\frac{B^{2}}{4 A}}\right)}{\partial B}=\sqrt{\frac{\pi}{A}} \frac{e^{\frac{B^{2}}{4 A}}}{2 A}\left(1+\frac{B^{2}}{2 A}\right) \tag{11}
\end{align*}
$$

Plugging in $B=0$ again, we obtain the following:

$$
\begin{equation*}
\int_{-\infty}^{\infty} x^{2} e^{-A x^{2}} d x=\sqrt{\frac{\pi}{A}} \frac{1}{2 A} \tag{12}
\end{equation*}
$$

So, we can continue this process.
In other words,

$$
\begin{equation*}
\frac{\partial^{p} \int_{-\infty}^{\infty} e^{-A x^{2}+B x} d x}{\partial B^{p}}=\int_{-\infty}^{\infty} x^{p} e^{-A x^{2}+B x} d x \tag{13}
\end{equation*}
$$

## Summary

- The following expression can be evaluated by completing the square the exponents

$$
\int_{-\infty}^{\infty} e^{-A x^{2}+B x} d x
$$

- The following expression

$$
\int_{-\infty}^{\infty} x^{p} e^{-A x^{2}} d x
$$

can be evaluated by using

$$
\frac{\partial^{p} \int_{-\infty}^{\infty} e^{-A x^{2}+B x} d x}{\partial B^{p}}=\int_{-\infty}^{\infty} x^{p} e^{-A x^{2}+B x} d x
$$

and plugging $B=0$.

