A short introduction to quantum mechanics X: position and momentum basis and Fourier transformation

We have already noted that P, the momentum operator, acts by $-i\hbar\frac{\partial}{\partial x}$. (Some textbooks use $\hat{p}$ for the momentum operator.) In other words, if we have:

$$|\psi\rangle = \int dx|x\rangle\langle x|\psi\rangle = \int \psi(x)|x\rangle dx \tag{1}$$

then it follows that:

$$P|\psi\rangle = \int \left(-i\hbar\frac{\partial\psi(x)}{\partial x}\right)|x\rangle dx = \int dx|x\rangle\langle x|P|\psi\rangle \tag{2}$$

Therefore, we have:

$$-i\hbar\frac{\partial}{\partial x}\langle x|\psi\rangle = \langle x|P|\psi\rangle \tag{3}$$

Now, let's plug in $|p\rangle$, the eigenvector of the momentum matrix with eigenvalue p, for $|\psi\rangle$. We get:

$$\begin{aligned} -i\hbar\frac{\partial}{\partial x}\langle x|p\rangle &= \langle x|P|p\rangle \\ -i\hbar\frac{\partial}{\partial x}\langle x|p\rangle &= \langle x|p|p\rangle \\ -i\hbar\frac{\partial}{\partial x}\langle x|p\rangle &= p\langle x|p\rangle \end{aligned} \tag{4}$$

To transition from the second line to the third line, we used the fact that p is merely a number, since it's an eigenvalue.

So, this is a differential equation for $\langle x|p\rangle$ and the solution is given by:

$$\langle x|p\rangle = Ce^{ipx/\hbar} \tag{5}$$

Then we can say:

$$|p\rangle = \int dx|x\rangle\langle x|p\rangle = \int dx|x\rangle Ce^{ipx/\hbar} \tag{6}$$

Given this, I will now explain the relation between wave functions written in the position basis and those written in the momentum basis. We may write a wave function $|\psi\rangle$ as follows:

$$|\psi\rangle = \int dx|x\rangle\langle x|\psi\rangle = \int dx\psi(x)|x\rangle \tag{7}$$

In the momentum basis, the same wave function $|\psi\rangle$ can be expressed as follows:

$$|\psi\rangle = \int dp|p\rangle\langle p|\psi\rangle = \int dp\phi(p)|p\rangle \tag{8}$$

where $\phi(p) = \langle p|\psi\rangle$

Now, observe:

$$\langle x|\psi\rangle = \int \langle x|p\rangle\langle p|\psi\rangle dp \tag{9}$$

$$\psi(x) = \int C\phi(p)e^{ipx/\hbar}dp \tag{10}$$

Given the above formula, could we express $\phi(p)$ in terms of $\psi(x)$? If you are careful enough, you will see that it is just a Fourier transformation problem. So, roughly speaking, we should have, $\phi(p) \sim \int \psi(x)e^{-ipx/\hbar}dp$ with an overall factor to be determined. Remarkably, it is possible to reproduce this as follows:

$$\langle p|\psi\rangle = \int \langle p|x\rangle\langle x|\psi\rangle dx \tag{11}$$

$$\phi(p) = \int C^*\psi(x)e^{-ipx/\hbar}dx \tag{12}$$

where we have used the fact that $\langle p|x\rangle = \langle x|p\rangle^*$. So, we have reproduced the so-called inverse Fourier transformation as advertised!

By comparing (10) and (12) with (6) and (7) of my article "Fourier transformations" and assuming that C is real without a loss of generality, we can get $C = 1/(\sqrt{2\pi\hbar})$. The fact that $\langle x|p\rangle$ is proportional to $e^{ipx/\hbar}$ makes sense, as it leads to the correct Fourier transformations; we see that quantum mechanics is mathematically consistent. If you study physics and math further, you will frequently encounter such beautiful consistencies!

To summarize, we derived:

$$\psi(x) = \int \frac{dp}{\sqrt{2\pi\hbar}}\phi(p)e^{ipx/\hbar} \tag{13}$$

$$\phi(p) = \int \frac{dx}{\sqrt{2\pi\hbar}}\psi(x)e^{-ipx/\hbar} \tag{14}$$

So far, in position basis, we have seen that X acts by multiplying the position-space wave function by x and P acts by $-i\hbar\frac{\partial}{\partial x}$ where the position-space wave function for a state $|\psi\rangle$ is given by $\langle x|\psi\rangle$. Then, a natural question to ask is how X and P act on the momentum-space wave function $\langle p|\psi\rangle$.

How P acts is easy. We have:

$$\langle p|P|\psi\rangle = (\langle p|P)|\psi\rangle = p\langle p|\psi\rangle \tag{15}$$

where we have used $P|p\rangle = p|p\rangle$. In other words, P acts by multiplying by p.

To figure out how X acts, let's consider the following equation. (**Problem 1.** Prove this! Hint: Insert $I = \int dx|x\rangle\langle x|$ between e^{iaX} and $|p\rangle$)

$$e^{iaX}|p\rangle = |p + \hbar a\rangle \tag{16}$$

For infinitesimal $a = \Delta p/\hbar$, the above equation implies

$$\begin{aligned} e^{i\Delta pX/\hbar}|p\rangle &= |p + \Delta p\rangle \\ (1 + \frac{i\Delta pX}{\hbar})|p\rangle &= |p + \Delta p\rangle \\ \frac{i\Delta pX}{\hbar}|p\rangle &= |p + \Delta p\rangle - |p\rangle \end{aligned} \tag{17}$$

Given this, we have:

$$\begin{aligned} |\psi\rangle &= \int dp|p\rangle\langle p|\psi\rangle = \int dp|p\rangle\psi(p) \\ \frac{i\Delta pX}{\hbar}|\psi\rangle &= \int dp\frac{i\Delta pX}{\hbar}|p\rangle\langle p|\psi\rangle \\ &= \int dp(|p + \Delta p\rangle - |p\rangle)\langle p|\psi\rangle \\ &= \int dp|p + \Delta p\rangle\langle p|\psi\rangle - \int dp|p\rangle\langle p|\psi\rangle \\ &= \int dp|p\rangle\langle p - \Delta p|\psi\rangle - \int dp|p\rangle\langle p|\psi\rangle \\ &= \int dp(\psi(p - \Delta p) - \psi(p))|p\rangle \\ \frac{i\Delta pX}{\hbar}|\psi\rangle &= \int dp\left(-\frac{\partial\psi}{\partial p}\Delta p\right)|p\rangle \\ X|\psi\rangle &= \int dp\left(i\hbar\frac{\partial\psi}{\partial p}\right)|p\rangle \end{aligned} \tag{18}$$

Therefore, we see that, in the momentum basis, the position operator X acts by $i\hbar\frac{\partial}{\partial p}$. Summarizing, we have:

$$X\psi(p) = i\hbar\frac{\partial\psi(p)}{\partial p} \tag{19}$$

$$P\psi(p) = p\psi(p) \tag{20}$$

where $\psi(p) = \langle p|\psi\rangle$.

Problem 2. Check $XP - PX = i\hbar$ in momentum basis.

Compare (19) and (20) with the following formulas for the familiar position basis:

$$X\psi(x) = x\psi(x) \tag{21}$$

$$P\psi(x) = -i\hbar\frac{\partial\psi(x)}{\partial x} \tag{22}$$

where $\psi(x) = \langle x|\psi\rangle$. We see that there is a symmetry upon simultaneous exchange of x for p and of i for $-i$.

Finally, let me conclude this article with some observations on Dirac delta function. Changing the integration variable from x to x' in (14), and plugging this to (13), we obtain:

$$\psi(x) = \int dx'\psi(x')\left(\int \frac{dp}{2\pi\hbar}e^{ip(x-x')/\hbar}\right) \tag{23}$$

Therefore, we conclude:

$$\delta(x - x') = \int \frac{dp}{2\pi\hbar}e^{ip(x-x')/\hbar} \tag{24}$$

Therefore, we see that the condition that one must come back to the original function, if one Fourier-transform and inverse-Fourier-transform it, yields a formula for Dirac delta function.

In other words, we derived an explicit formula for delta function as follows:

$$\delta(x) = \int \frac{dp}{2\pi}e^{ipx} \tag{25}$$

Actually, we can derive this equation more easily as follows:

$$\begin{aligned}\delta(x - x') &= \langle x|x'\rangle = \int dp\langle x|p\rangle\langle p|x'\rangle \\ &= \int \frac{dp}{2\pi\hbar}e^{ip(x-x')/\hbar}\end{aligned} \tag{26}$$

Now more problems. Consider a wave function $\phi(x)$ as follows:

$$\phi(x) = C_1 \exp\left[ikx - \frac{x^2}{2d^2}\right] \tag{27}$$

Problem 3. Use Taylor series to show

$$e^{iPa/\hbar}\psi(x) = \psi(x + a) \tag{28}$$

In other words, the momentum operator P moves the position of the wave function by a certain amount. Thus, we say P generates "translation" (i.e.

the movement of position). In the next article, we will see that the Hamiltonian operator H moves the time of the wave function. In other words, we say H generates "time evolution."

Problem 4. Show that the probability density function for position due to (27) is given by a normal (i.e. a Gaussian) distribution. (Hint: See our earlier article "Probability density function.") Thus, we call such a wave "Gaussian wave packet."

Problem 5. Obtain, C_1 by assuming that $\phi(x)$ is properly normalized.

Problem 6. Find Δx, the standard deviation of the position x, in terms of d. Find the expectation value of the momentum p without Fourier-transforming $\phi(x)$. Find the expectation value of p^2 similarly. Thus, obtain Δp, the standard deviation of the momentum.

Problem 7. Fourier transform $\phi(x)$ to obtain $\psi(p)$, the wave function in momentum space. Thus, show that the probability density function in the momentum space is also given by a normal distribution. From this, obtain the expectation value of p and the standard deviation Δp again. Check that they agree with the values you obtained using the wave function in position basis. Check also that following holds.

$$\Delta x \Delta p = \frac{\hbar}{2} \tag{29}$$

This, you found in case of Gaussian packet. Generally, the following holds:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{30}$$

This is called "Heisenberg's uncertainty principle." We will talk more about this in a later article.

Summary

- $\langle x|p\rangle$ is proportional to $e^{ipx/\hbar}$.
- The relation between position basis and momentum basis is that of Fourier transformation and inverse Fourier transformation.