A short introduction to quantum mechanics XII: Heisenberg's uncertainty principle

Heisenberg's uncertainty principle asserts that the more precisely you know an object's position, the less precisely you know its momentum and vice versa. This fact can be derived rigorously by mathematics. In this article, we explain Heisenberg's uncertainty principle and mathematically derive it.

Suppose an object traveling with a certain, exact, velocity in the positive x-direction. In other words, it is traveling with a certain, exact momentum. This would imply that the wave function of the object is the eigenstate of the momentum operator. If you recall what we have discussed in our last article, you will see that the wave function in this case should be given as follows:

$$\psi = C(\cos(ipx/\hbar) + i\sin(ipx/\hbar)) \tag{1}$$

Let's draw this wave function. See Fig. 1. For simplicity, we have only drawn the imaginary part of the wave function. So, the momentum of the object has a certain, exact value. On the other hand, it would be hard to locate its position. Where is it located? At $x = -6$? $x = -3$? $x = 0$? $x = 4$? There is no fixed location, and it has actually equal probability to be found anywhere! As just stated in the beginning of the article, as we know the object's momentum very precisely, we cannot locate its position precisely. Heisenberg's uncertainty principle is mathematically stated as follows:

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}, \quad \Delta y \Delta p_y \geq \frac{\hbar}{2}, \quad \Delta z \Delta p_z \geq \frac{\hbar}{2}, \quad \Delta E \Delta t \geq \frac{\hbar}{2} \tag{2}$$

where Δx is the standard deviation of the x-position, and Δp_x is the standard deviation of the x-momentum and so on.

In our case for Fig. 1. $\Delta p_x = 0$, which forces $\Delta x = \infty$ from above equation. On the other hand, see Fig. 2, which is somewhat the opposite case of Fig. 1. The object is well-located at $x = 0$ with Δx roughly being around 1. However, the value for the wavelength cannot be quite well-determined, since there are only $3 \sim 4$ oscillations. So, the standard deviation of the wavelength is big. (After all, if it were zero, we would have had Fig. 1.) Since the wavelength gives the value for the momentum by the de-Broglie

Fig. 1

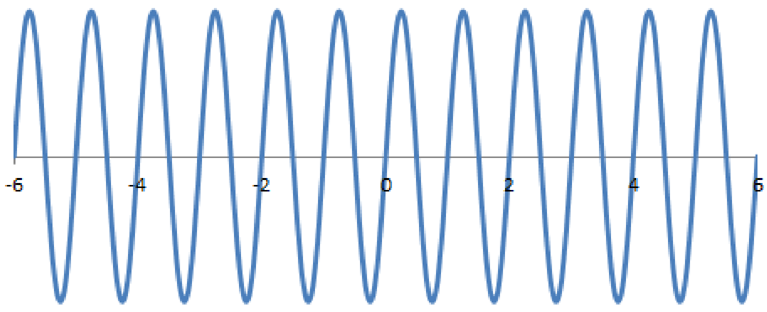


Fig. 2

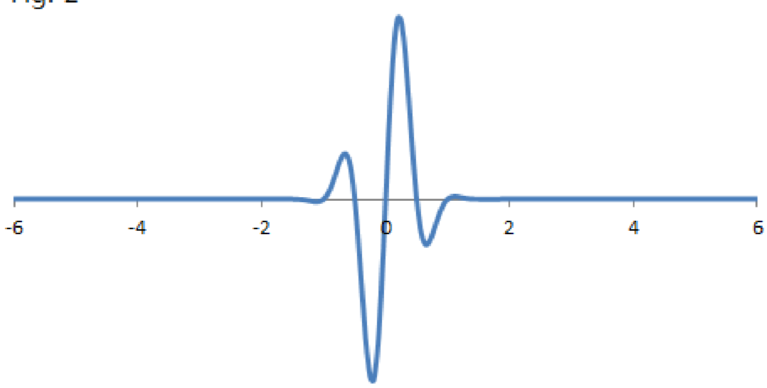
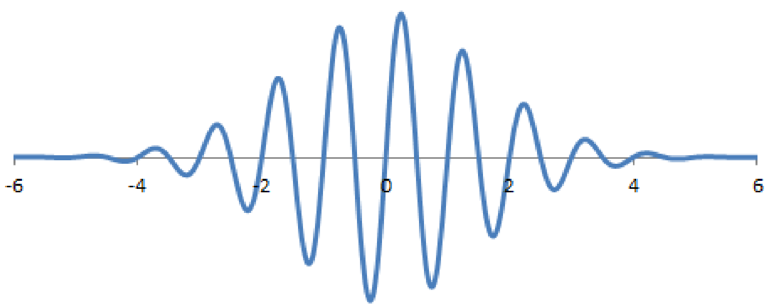


Fig. 3



formula, ($p = h/\lambda$) we can say that the standard deviation of the momentum is large. Again, we recover our assertion that the more precisely you know an object's location, the less precisely you know its momentum.

For just one more example, we have our third case Fig. 3. Its Δx is smaller than that of Fig. 1 while bigger than that of Fig. 3. Therefore, according to (2), it is allowed that Fig. 2's Δp_x is bigger than that of Fig. 1 while smaller than that of Fig. 3. This is actually the case. Fig. 3 looks closer to Fig. 1 than Fig. 2 does. Therefore, Fig. 2's Δx and Δp_x are between those of Fig. 1 and Fig. 2.

Now, let's derive Heisenberg's uncertainty principle. To this end, we must first recall Cauchy-Schwarz inequality. In one of the exercises in our earlier article "The dot product," you essentially proved that Cauchy-Schwarz inequality can be written as

$$|\vec{u}|^2|\vec{v}|^2 \geq (\vec{u} \cdot \vec{v})^2 \tag{3}$$

In case of complex vector space, this becomes

$$|\vec{u}|^2|\vec{v}|^2 \geq |\vec{u} \cdot \vec{v}|^2 \tag{4}$$

Otherwise, the right-hand side of (3) can be a number that is not real, which is problematic, as we cannot compare whether a number is bigger or smaller for an imaginary number.

Problem 1. Let's denote the expectation value of an operator O by $\langle O \rangle \equiv \langle \psi | O | \psi \rangle$. Let A and B two Hermitian operators. By plugging in $\vec{u} = A|\psi\rangle$ and $\vec{v} = B|\psi\rangle$ into (4), show that

$$(\langle A \rangle \langle B \rangle)^2 \geq |\langle AB \rangle|^2 \tag{5}$$

Problem 2. Let $\langle AB \rangle \equiv \langle \psi | AB | \psi \rangle = f + gi$. Then, show the following:

$$\langle BA \rangle \equiv \langle \psi | BA | \psi \rangle = f - gi \tag{6}$$

$$\frac{1}{2}\langle [A, B] \rangle \equiv \frac{1}{2}\langle \psi | [A, B] | \psi \rangle = gi \tag{7}$$

Problem 3. Show that the right-hand side of (5) is given by

$$|\langle AB \rangle|^2 = f^2 + g^2 \tag{8}$$

Problem 4. By combining (7) and (8), show that

$$|\langle AB \rangle|^2 \geq \frac{1}{4}|\langle [A, B] \rangle|^2 \tag{9}$$

Thus, by combining (5) and (9), we obtain

$$(\langle A \rangle \langle B \rangle)^2 \geq \frac{1}{4}|\langle [A, B] \rangle|^2 \tag{10}$$

Now, let

$$A = \Delta x \equiv x - \langle x \rangle, \quad B = \Delta p_x \equiv p_x - \langle p_x \rangle \tag{11}$$

since $\langle x \rangle$ and $\langle p_x \rangle$ are mere numbers, we have:

$$[A, B] = [x - \langle x \rangle, p_x - \langle p_x \rangle] = [x, p] = i\hbar \tag{12}$$

Problem 6. By plugging in (11) and (12) into (10), show that

$$\langle \Delta x \rangle \langle \Delta p_x \rangle \geq \frac{\hbar}{2} \tag{13}$$

In other words, we just proved the first relation in (2). We can prove the second and the third relations similarly, from $[y, p_y] = [z, p_z] = i\hbar$. However, we cannot prove $\Delta E \Delta t \geq \hbar/2$ using this method; t is just a coordinate, not an operator. Nevertheless, it sounds reasonable, if you look at this from the point of view of Fourier transformation. Let me clarify what I mean. If we write

$$\psi(x, y, z, t) = \int \frac{dE dp_x dp_y dp_z}{(2\pi\hbar)^2} \phi(E, p_x, p_y, p_z) e^{i(p_x x + p_y y + p_z z - Et)/\hbar} \tag{14}$$

then, Heisenberg's uncertainty principle says that there are following relations between ψ and ϕ.

$$\left(\int x^2 \psi\psi^* dx - \left(\int x\psi\psi^* dx \right)^2 \right) \left(\int p_x^2 \phi\phi^* dp_x - \left(\int p_x \phi\phi^* dp_x \right)^2 \right) \geq \frac{\hbar^2}{4}$$

In other words, this is a relation that says about a property of Fourier transformation. As this property must be satisfied for other conjugate variables for Fourier transformation, we can write

$$\left(\int t^2 \psi\psi^* dt - \left(\int t\psi\psi^* dt \right)^2 \right) \left(\int E^2 \phi\phi^* dE - \left(\int E \phi\phi^* dE \right)^2 \right) \geq \frac{\hbar^2}{4}$$

which implies $\Delta E \Delta t \geq \hbar/2$.

Final comment. Notice that plugging $A = \Delta y$, $B = \Delta p_z$ into (10) yields:

$$\Delta y \Delta p_z \geq 0 \tag{15}$$

Therefore, if the component measured for the position and the one for the momentum are different, there is no restriction for uncertainty; knowing the y-position of an object doesn't hinder from knowing its z-momentum precisely. Similarly, from $[x, y] = [x, z] = [y, z] = 0$, we have

$$\Delta x \Delta y \geq 0, \qquad \Delta x \Delta z \geq 0, \qquad \Delta y \Delta z \geq 0 \tag{16}$$

Therefore, we see that knowing the exact x-position of a particle doesn't hinder from knowing its exact y-position or its exact z-position and vice versa. Thus, there can be a state that has a certain x-position, a certain y-position, and a certain z-position (i.e., $\Delta x = \Delta y = \Delta z = 0$) at the same time. This state is the eigenstate (eigenvector) of the x-position operator, the y-position operator and the z-position operator. If we denote this state by $|x, y, z\rangle$, we have

$$\hat{x}|x, y, z\rangle = x|x, y, z\rangle, \quad \hat{y}|x, y, z\rangle = y|x, y, z\rangle, \quad \hat{z}|x, y, z\rangle = z|x, y, z\rangle \tag{17}$$

What we just said is true for the momentum. As $[p_x, p_y] = [p_x, p_z] = [p_y, p_z] = 0$, we have

$$\Delta p_x \Delta p_y \geq 0, \qquad \Delta p_x \Delta p_z \geq 0, \qquad \Delta p_y \Delta p_z \geq 0 \tag{18}$$

and we can have eigenstates of $\hat{p}_x$, $\hat{p}_y$ and $\hat{p}_z$ at the same time as follows.

$$\hat{p}_x|p_x, p_y, p_z\rangle = p_x|p_x, p_y, p_z\rangle, \quad \hat{p}_y|p_x, p_y, p_z\rangle = p_y|p_x, p_y, p_z\rangle$$
$$\hat{p}_z|p_x, p_y, p_z\rangle = p_z|p_x, p_y, p_z\rangle \tag{19}$$

However, we cannot have a state that is an eigenstate of $\hat{x}$ and $\hat{p}_x$ at the same time. An eigenstate of $\hat{x}$ satisfies $\langle \Delta x \rangle = 0$ and an eigenstate of $\hat{p}_x$ satisfies $\langle \Delta p_x \rangle = 0$. If a state is an eigenstate of $\hat{x}$ and $\hat{p}_x$ at the same time, it violates Heisenberg's uncertainty principle, as $\Delta x \Delta p_x$ would be zero. It is not hard to see this, even if you didn't know Heisenberg's uncertainty principle. Suppose there is such an eigenstate $|x, p_x\rangle$. Then, we have

$$\hat{x}|x, p_x\rangle = x|x, p_x\rangle, \qquad \hat{p}_x|x, p_x\rangle = p_x|x, p_x\rangle \tag{20}$$

which implies

$$\hat{x}\hat{p}_x|x, p_x\rangle = \hat{x}p_x|x, p_x\rangle = p_x\hat{x}|x, p_x\rangle = p_x x|x, p_x\rangle = xp_x|x, p_x\rangle \tag{21}$$

where we used the fact that p_x and x are numbers, not operators. Similarly,

$$\hat{p}_x\hat{x}|x, p_x\rangle = \hat{p}_x x|x, p_x\rangle = x\hat{p}_x|x, p_x\rangle = xp_x|x, p_x\rangle \tag{22}$$

Combining (21) and (22), we obtain

$$[\hat{x}, \hat{p}_x]|x, p_x\rangle = \hat{x}\hat{p}_x|x, p_x\rangle - \hat{p}_x\hat{x}|x, p_x\rangle \tag{23}$$
$$= xp_x|\psi\rangle - xp_x|\psi\rangle = 0 \tag{24}$$

However, the left-hand side of (23) is $i\hbar|x, p_x\rangle$. Thus, we obtain

$$i\hbar|x, p_x\rangle = 0 \tag{25}$$

In other words, there is no state that is an eigenstate of x and p_x at the same time.

Anyhow, we have seen that there can be a state that has a fixed 3-dimensional position and a state that has a fixed 3-dimensional momentum. Then, you may ask, can there be a state that has a fixed 3-dimensional angular momentum? We will answer this question in "Angular momentum in quantum mechanics."

Summary

- Heisenberg's uncertainty relation is given by

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}, \quad \Delta y \Delta p_y \geq \frac{\hbar}{2}, \quad \Delta z \Delta p_z \geq \frac{\hbar}{2}, \quad \Delta E \Delta t \geq \frac{\hbar}{2}$$

- It can be derived using Cauchy-Schwarz inequality.