A short introduction to quantum mechanics I addendum: the normalization

If you read our last article, you will see that the probabilities that a particle with the following state vector

$$|\psi\rangle = 7|2\mathrm{J}\rangle - |5\mathrm{J}\rangle \tag{1}$$

will have 2J of energy and 5J of energy is 49/50 and 1/50 respectively. How did you calculate? Like this:

$$\frac{7^2}{7^2+(-1)^2}, \qquad \frac{(-1)^2}{7^2+(-1)^2} \tag{2}$$

Actually, there are couple of things that I didn't mention in the last article: that the norm of $|2\mathrm{J}\rangle$ and $|5\mathrm{J}\rangle$ are both 1 and they are orthogonal to each other, i.e., $\langle 2\mathrm{J}|5\mathrm{J}\rangle = 0$. If you think about it, we often had similar conditions in our 2 or 3-dimensional vector space. For example,

$$\langle 2\mathrm{J}|2\mathrm{J}\rangle = 1 \ \text{ is similar to } \ \hat{x}\cdot\hat{x} = 1 \tag{3}$$

$$\langle 5\mathrm{J}|5\mathrm{J}\rangle = 1 \ \text{ is similar to } \ \hat{y}\cdot\hat{y} = 1 \tag{4}$$

$$\langle 2\mathrm{J}|5\mathrm{J}\rangle = 0 \ \text{ is similar to } \ \hat{x}\cdot\hat{y} = 0 \tag{5}$$

Then, the norm of $|\psi\rangle$ can be calculated as follows:

$$\begin{aligned}
\langle\psi|\psi\rangle &= (7\langle 2\mathrm{J}| - \langle 5\mathrm{J}|)\,(7|2\mathrm{J}\rangle - |5\mathrm{J}\rangle) && (6)\\
&= 7^2\langle 2\mathrm{J}|2\mathrm{J}\rangle + (-1)^2\langle 5\mathrm{J}|5\mathrm{J}\rangle - 7\langle 2\mathrm{J}|5\mathrm{J}\rangle - 7\langle 5\mathrm{J}|2\mathrm{J}\rangle && (7)\\
&= 49+1+0+0 = 50 && (8)
\end{aligned}$$

So, the norm is $\sqrt{50}$.

Now, notice this. As we will see shortly, there is a quicker way to calculate the probabilities if we consider the following state vector $|\tilde{\psi}\rangle$ instead of $|\psi\rangle$ in (1).

$$|\tilde{\psi}\rangle = \frac{|\psi\rangle}{\sqrt{50}} = \frac{7}{\sqrt{50}}|2\mathrm{J}\rangle - \frac{1}{\sqrt{50}}|5\mathrm{J}\rangle \tag{9}$$

Two things to note. First, the norm of $|\tilde{\psi}\rangle$ is 1, because the norm of $|\psi\rangle$ is $\sqrt{50}$ and $|\tilde{\psi}\rangle$ is defined by $|\psi\rangle$ divided by $\sqrt{50}$. Second, it is easier to calculate the probabilities. They are simply given by the square of coefficients as follows:

$$\left(\frac{7}{\sqrt{50}}\right)^2 = \frac{49}{50}, \qquad \left(\frac{-1}{\sqrt{50}}\right)^2 = \frac{1}{50} \tag{10}$$

Notice that the condition that the norm of $|\tilde{\psi}\rangle$ is 1 guaranties that the probabilities that we just calculated add up to 1 as follows:

$$\langle\tilde{\psi}|\tilde{\psi}\rangle = \left(\frac{7}{\sqrt{50}}\right)^2 + \left(\frac{-1}{\sqrt{50}}\right)^2 = \frac{49}{50} + \frac{1}{50} = 1 \tag{11}$$

Therefore, as far as the state vector is concerned, physicists consider a vector like (9) instead of the one like (1), because of the convenience of the former. After all, it is not hard to transform a vector like (1) to a vector like (9). We can just divide the vector by its norm. Then, the norm of the new vector is 1. Such a process, i.e., making the norm of vector 1, is called "normalization." After all, the vector before the normalization and the vector after the normalization correspond to the same state of particle they describe, as they always yield the same probabilities that something happens. Normalization is a mathematical process rather than a physical one.

There are couple of things that I forgot to mention. If you read our earlier article on complex vector space, you will remember that the coefficients can be complex numbers. Thus, if the probabilities are given by the coefficients squared they can also be complex numbers. This is troublesome, as probabilities must always be non-negative real numbers. The remedy is simple. The probability is given by the square of its magnitude instead of the square.

Problem 1. Let $\langle 2\mathrm{J}|2\mathrm{J}\rangle = \langle 3\mathrm{J}|3\mathrm{J}\rangle = 1$ and $\langle 2\mathrm{J}|3\mathrm{J}\rangle = 0$. Then, show that the following $|\psi\rangle$ is normalized.

$$|\psi\rangle = \left(\frac{1-i}{\sqrt{6}}\right)|2\mathrm{J}\rangle + \left(\frac{2}{\sqrt{6}}\right)|3\mathrm{J}\rangle \tag{12}$$

Problem 2. If a wave function of a particle is given by the above state vector, what will be the probability of an observer observing its energy to be 2J? What about 3J? Check that the total probability adds up to 1.

So far, we have talked about the normalization of state vector. Let's now talk about the normalization of eigenvector such as $|2\mathrm{J}\rangle$, $|3\mathrm{J}\rangle$ and $5\mathrm{J}\rangle$ in our examples. Their normalization is as simple as the one of state vector. Once you obtain an eigenvector you can divide it by its norm to obtain the normalized eigenvector. If you are not sure what I mean, re-read "Eigenvalues and eigenvectors" where I mentioned this.

So, we just talked about (3) and (4), i.e., the normalization of eigenvectors. What we haven't talked about is (5). Notice that (5) cannot be achieved by normalization as in (3) and (4). If two vectors are not orthogonal, the two vectors will not be still orthogonal even if you change their length as long as you do not change their direction. We will talk about why (5) is satisfied in our later article on Hermitian matrices.

Summary

- If we make the norm of a vector 1 by dividing the original vector by its norm, we call it "normalization."
- A normalized state vector is convenient to calculate probabilities, so physicists usually consider normalized state vectors instead of the un-normalized one.
- The probability of observing a given eigenvalue is given by the square of the absolute value of the coefficient of the corresponding eigenvector. In this case, the eigenvectors must be normalized.