A short introduction to quantum mechanics III: the equivalence between Heisenberg's matrix method and Schrödinger's differential equation

In the article "A short introduction to quantum mechanics I: observables and eigenvalues," I explained that Heisenberg's quantum mechanics is based on the formula $XP - PX = i\hbar$, where X is the position operator and P is the momentum operator, while Schrödinger's quantum mechanics is based on the idea that the energy matrix is $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$. I claimed, without proof, that the two formalisms are equivalent. In this article, I will concretely show that they are indeed equivalent.

The key idea to understanding this is that $XP - PX = i\hbar$ can be satisfied if the position operator X corresponds to multiplying the wave function by x, while the momentum operator P corresponds to $-i\hbar\frac{\partial}{\partial x}$(differentiating with respect to x and multiplying by $-i\hbar$). Now, let's see how this corresponds to Heisenberg's quantum mechanics. If we apply the momentum operator P to the vector $\psi(x)$, we get $P\psi(x) = -i\hbar\frac{\partial\psi(x)}{\partial x}$. If we then apply the position operator X to this, we get $-i\hbar x\frac{\partial\psi(x)}{\partial x}$. In other words:

$$XP\psi(x) = -i\hbar x\frac{\partial\psi(x)}{\partial x} \tag{1}$$

Similarly we can easily obtain

$$X\psi(x) = x\psi(x) \tag{2}$$

$$PX\psi(x) = P(X\psi(x)) = -i\hbar\frac{\partial(x\psi(x))}{\partial x} = -i\hbar\left(\psi(x) + x\frac{\partial\psi(x)}{\partial x}\right) \tag{3}$$

One more step forward, we get:

$$(XP - PX)\psi(x) = i\hbar\psi(x) \tag{4}$$

In other words, $XP - PX = i\hbar$. This is Heisenberg's matrix method. Indeed the condition $XP - PX = i\hbar$ is equal to the condition that the position operator X corresponds to multiplying the wave function by x and the momentum operator P corresponds to $-i\hbar\frac{\partial}{\partial x}$.

Now, let's derive Schrödinger's equation. In classical mechanics, mechanical energy is

$$E = \frac{1}{2}mv^2 + V(x) = \frac{(mv)^2}{2m} + V(x) = \frac{p^2}{2m} + V(x) \tag{5}$$

Putting this into the language of operators, p^2 means applying P twice to the vector $\psi(x)$, while $V(x)$ means multiplying $\psi(x)$ by $V(x)$. In other words, p^2 is $-\hbar^2\frac{\partial^2}{\partial x^2}$ If we divide this by $2m$ and add $V(x)$ we obtain the energy matrix. If we then apply the energy matrix to the vector $\psi(x)$, we get:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) \tag{6}$$

We can get the eigenvalues and the eigenvectors of this energy matrix by solving the equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x) \tag{7}$$

where E is the eigenvalue.

At this point, we would like to introduce commutator. A commutator of A and B is defined by $AB - BA$ and denoted as $[A, B]$, For example, our earlier formula can be re-written as follows:

$$[X, P] = i\hbar \tag{8}$$

Problem 1. Prove the followings.

$$[A, B] = -[B, A], \qquad [A, A] = 0 \tag{9}$$

$$[A, B + C] = [A, B] + [A, C], \qquad [A + B, C] = [A, C] + [B, C] \tag{10}$$

$$[A + B, C + D] = [A, C] + [B, C] + [A, D] + [B, D] \tag{11}$$

$$[cA, dB] = cd[A, B], \qquad \text{where } c \text{ and } d \text{ are numbers} \tag{12}$$

Problem 2. Use (11) and (12) to prove the following.

$$[A + Bi, A - Bi] = i[B, A] - i[A, B] = 2i[B, A] \tag{13}$$

Problem 3. Prove the followings.

$$[AB, C] = A[B, C] + [A, C]B \tag{14}$$

$$[D, EF] = [D, E]F + E[D, F] \tag{15}$$

Problem 4. Using (14) and (15), prove the followings:

$$[X^2, P_x] = 2i\hbar X \tag{16}$$

$$[X, P_x^2] = 2i\hbar P_x \tag{17}$$

Problem 5. Using Leibniz rule and $P\psi(x) = -i\hbar\frac{\partial\psi(x)}{\partial x}$, prove the following:

$$[f(X), P_x] = i\hbar\frac{\partial f(x)}{\partial x} \tag{18}$$

(Hint[1]) Notice that we could have obtained (16) using the above formula.

Summary

- A commutator of A and B is defined by $AB - BA$ and denoted as $[A, B]$.

[1]Show $[f(X), P_x]\psi = i\hbar\frac{\partial f(x)}{\partial x}\psi$

- $[X, P] = i\hbar$.
- The position operator X acts by multiplying x.
- The momentum operator P_x acts by $-i\hbar\frac{\partial}{\partial x}$.
- $[A, B] = -[B, A], \qquad [A, A] = 0$
- $[AB, C] = A[B, C] + [A, C]B$
- $[D, EF] = [D, E]F + E[D, F]$