A short introduction to quantum mechanics IV addendum: revisiting the normalization

Let's consider a state vector given by $|\psi\rangle$. Then, what is the probability that we obtain certain values for observable A when we measure it? If $\hat{A}$ is the linear operator that corresponds to A, we need to find the eigenvalues and the eigenvectors of $\hat{A}$. Let's say that the eigenvalue is λ_n and its corresponding normalized eigenvector $|\lambda_n\rangle$. In other words,

$$\langle\lambda_n|\lambda_m\rangle = \delta_{nm} \tag{1}$$

Then, we can express a state vector as a linear combination of these eigenvectors as follows.

$$|\psi\rangle = \sum_n c_n|\lambda_n\rangle \tag{2}$$

As we explained earlier it is convenient if $|\psi\rangle$ is normalized. So, let's assume that it is normalized; even if it wasn't, we can always normalize it. Then,

$$1 = \langle\psi|\psi\rangle = \left(\sum_m c_m^*\langle\lambda_m|\right)\left(\sum_n c_n|\lambda_n\rangle\right) \tag{3}$$

$$1 = \sum_m\sum_n c_m^* c_n\langle\lambda_m|\lambda_n\rangle \tag{4}$$

$$1 = \sum_m\sum_n c_m^* c_n\delta_{mn} \tag{5}$$

$$1 = \sum_m c_m^* c_m = \sum_m |c_m|^2 \tag{6}$$

I explained earlier that $|c_m|^2$ is the probability that we will obtain λ_m for the observable A. In other words, (6) means that the total probability is 1.

Then, how can we obtain c_m? Recall our earlier article on Dirac's bra-ket notation. If we have

$$|v\rangle = \sum_i v_i|e_i\rangle, \qquad \langle e_j|e_i\rangle = \delta_{ji} \tag{7}$$

we have

$$\langle e_j|\vec{v}\rangle = \sum_i v_i\langle e_j|e_i\rangle = \sum_i v_i\delta_{ji} = v_j \tag{8}$$

For example, if we have $\vec{v} = 2\hat{x} + 3\hat{y} - 4\hat{z}$, the z component of $\vec{v}$ is given by $\hat{z}\cdot\vec{v}$, which is -4.

Problem 1. Show the following from (1) and (2).

$$c_n = \langle\lambda_n|\psi\rangle \tag{9}$$

If we plug this c_n into (2), we obtain

$$|\psi\rangle = \sum_n \langle\lambda_n|\psi\rangle|\lambda_n\rangle = \sum_n |\lambda_n\rangle\langle\lambda_n|\psi \tag{10}$$

In other words, we obtain the following completeness relation.

$$I = \sum_n |\lambda_n\rangle\langle\lambda_n| \tag{11}$$

Now, let's go over to infinite-dimensional case.

$$I = \int_{-\infty}^{\infty} |x\rangle\langle x|dx \tag{12}$$

Thus,

$$|\psi\rangle = \int_{-\infty}^{\infty} \psi(x)|x\rangle dx \tag{13}$$

where

$$\langle x|\psi\rangle = \psi(x) \tag{14}$$

In other words, the above formula is just infinite-dimensional versions of (8) and (9).

How about the probabilities in this case? Considering that in the case of finite-dimensional case, the probability that we get λ_n is given by $|c_n|^2 = c_n^* c_n$ and $\psi(x)$ corresponds to c_n in infinite-dimensional case, the probability that a particle will be found at the position $a < x < b$에 is given by

$$P(a < x < b) = \int_a^b |\psi(x)|^2 \, dx = \int_a^b \psi^*(x)\psi(x)dx \tag{15}$$

Notice here that the probability is being represented by "summing" over all the squares of "coefficients" between x_a and x_b. In other words, the probability of finding the particle between x and $x + dx$ is given by $\phi^*(x)\phi(x)dx$.

Notice that the probability of finding a particle at the position between negative infinity and positive infinity is given by 1 implies

$$1 = \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx \tag{16}$$

In other words, the probability of finding a particle at anywhere is 1. As before, it is easy to see that this is the same condition that $|\psi\rangle$ is normalized as follows:

$$1 = \langle\psi|\psi\rangle = \langle\psi|1|\psi\rangle = \int_{-\infty}^{\infty} dx\langle\psi|x\rangle\langle x|\psi\rangle \tag{17}$$

Notice that (14) implies $\psi^*(x) = \langle\psi|x\rangle$. Thus, the above formula is equal to (16). This interpretation of the normalization of the state vector (that the probability sums up to 1) will play an important role when I discuss the unitarity of the time evolution operator in a later article.

Summary

- If a normalized state vector is given by

$$|\psi\rangle = \sum_n c_n |\lambda_n\rangle$$

where $|\lambda_n$s are normalized eigenvectors with eigenvalues of λ_n for Hermitian matrix $\hat{A}$ that corresponds to the observable A. The probability that we will get λ_n when we measure A is given by $|c_n|^2$.

- c_n can be obtained by the following formula:

$$c_n = \langle \lambda_n | \psi \rangle$$

- The probability that a particle will be found at the position $a < x < b$ is given by

$$P(a < x < b) = \int_a^b |\psi(x)|^2 \, dx = \int_a^b \psi^*(x)\psi(x) dx$$