A short introduction to quantum mechanics VI: position basis

The position operator is very different in nature than the energy operator. The value of a position of an object can take a continuous value. For example, it can be 0.4567 meter or $-3.112938475\cdots$ meter. Therefore, the position operator admits continuous eigenvalues. Given this, let $|x\rangle$ denote the eigenvector of the position matrix corresponding to eigenvalue x. In other words,

$$\hat{x}|x\rangle = x|x\rangle \tag{1}$$

where $\hat{x}$ is the position operator and x the eigenvalue. Using this notation, the completeness relation can be written as follows

$$1 = \int_{-\infty}^{\infty} dx|x\rangle\langle x| \tag{2}$$

as briefly mentioned in my article on Dirac's bra-ket notation. The integration range is from negative infinity to positive infinity since the position can take any value between these two numbers. Using this relation, for an arbitrary state vector $|\beta\rangle$, we have:

$$|\beta\rangle = \int_{-\infty}^{\infty} dx|x\rangle\langle x|\beta\rangle = \int_{-\infty}^{\infty} dx\langle x|\beta\rangle|x\rangle \tag{3}$$

where in the last step, we used the fact that a vector (i.e., $|x\rangle$) times a number (i.e., $\langle x|\beta\rangle$) is equal to the same number (i.e., $\langle x|\beta\rangle$) multiplied by the same vector (i.e., $|x\rangle$).

If we define $\beta(x) \equiv \langle x|\beta\rangle$, we have:

$$|\beta\rangle = \int_{-\infty}^{\infty} dx\beta(x)|x\rangle \tag{4}$$

In other words, $\beta(x)$ is the coefficient of the basis $|x\rangle$ for the vector $|\beta\rangle$.

This should be familiar from our earlier discussion in our earlier article. Namely,

$$|\psi\rangle = \sum_n c_n|\lambda_n\rangle, \qquad c_n = \langle\lambda_n|\psi\rangle \tag{5}$$

where c_n is the coefficient of the basis $|\lambda_n\rangle$ for the vector $|\psi\rangle$.

Similarly, for a ket-vector $\langle\alpha|$, we have:

$$\langle\alpha| = \int_{-\infty}^{\infty} dx \langle\alpha|x\rangle\langle x| = \int_{-\infty}^{\infty} dx \alpha^*(x)\langle x| \tag{6}$$

where we have used that $\langle\alpha|x\rangle = \langle x|\alpha\rangle^* = \alpha^*(x)$

Now, let's calculate the dot-product between α and β. Using (2), we obtain

$$\langle\alpha|\beta\rangle = \langle\alpha|1|\beta\rangle = \int_{-\infty}^{\infty} dx \langle\alpha|x\rangle\langle x|\beta\rangle = \int_{-\infty}^{\infty} dx \alpha^*(x)\beta(x) \tag{7}$$

This equation should remind you of the discussion from our fourth article on quantum mechanics. Namely, I explained that when $|A\rangle = \sum_i a_i|i\rangle$, and when $|B\rangle = \sum_i b_i|i\rangle$ where $|i\rangle$ is an orthonormal basis, we have $\langle A|B\rangle = \sum_i a_i^* b_i$. In our case, the sum is replaced by the integration, as the basis is now labeled by a continuous number x instead of the discrete index i.

In the last article, we explained how we can obtain the expectation value in quantum mechanics. So, how can obtain the expectation of the position x and the momentum p? It's easy. It is given by

$$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle, \qquad \langle p\rangle = \langle\psi|\hat{p}|\psi\rangle. \tag{8}$$

Let's calculate them. To this end, note

$$\hat{x}|\psi\rangle = \hat{x}\left(\int_{-\infty}^{\infty} \psi(x)|x\rangle\right) \tag{9}$$

where we used (4). Thus,

$$\hat{x}|\psi\rangle = \int_{-\infty}^{\infty} \psi(x)\hat{x}|x\rangle = \int_{-\infty}^{\infty} \psi(x)x|x\rangle \tag{10}$$

On the other hand,

$$\langle\psi| = \int_{-\infty}^{\infty} \psi^*(x)\langle x| \tag{11}$$

where we used (6). Thus, upon using (7), we get

$$\langle\psi|x|\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x)x\psi(x)dx = \int_{-\infty}^{\infty} x\psi^*(x)\psi(x)dx \tag{12}$$

Considering that $\psi^*(x)\psi(x)$ is the probability density, and x is the possible outcome, we see that this is indeed the correct formula for the expectation value of x.

Similarly,

$$\langle p\rangle = \langle\psi|\hat{p}|\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x)\left(-i\hbar\frac{\partial\psi(x)}{\partial x}\right)dx \tag{13}$$

Summary

- $|x\rangle$ is an eigenvector of position operator $\hat{x}$ with eigenvalue x.
- A state vector $|\psi\rangle$ can be expressed by its wave function $\psi(x)$ as

$$|\psi\rangle = \int_{-\infty}^{\infty} \psi(x)|x\rangle dx$$

  where $\psi(x) = \langle x|\psi\rangle$.

- The dot product of two state vectors $|u\rangle$ and $|v\rangle$ is given by

$$\langle u|v\rangle = \int_{-\infty}^{\infty} u^*(x)v(x)dx$$

- The expectation value of position is given by

$$\langle\psi|x|\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x)x\psi(x)dx = \int_{-\infty}^{\infty} x\psi^*(x)\psi(x)dx$$

- The expectation value of momentum is given by

$$\langle p\rangle = \langle\psi|\hat{p}|\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x)\left(-i\hbar\frac{\partial\psi(x)}{\partial x}\right)dx$$