A short introduction to quantum mechanics VII: the Hermiticity of the position operator and the momentum operator

In an earlier article, I noted that all the operators corresponding to observables should be Hermitian. In this article, we will explicitly verify that the position operator and the momentum operator satisfy this condition.

To this end, let's recall what Hermitian matrices were. Hermitian matrices are selfadjoint, meaning the complex conjugates of their transposes are equal to themselves. In component notation, this can be written as follows:

$$A_{ij} = A^*_{ji} \tag{1}$$

In Dirac's bra-ket notation, this can be written as

$$\langle\alpha|A|\beta\rangle = (\langle\beta|A|\alpha\rangle)^* \tag{2}$$

Now let's first consider the Hermiticity of the position operator. In position basis, the position operator X acts by multiplying x. (The position basis is the usual basis we have considered so far. You have to consult the next article to find out what position basis exactly means.) Therefore:

$$\langle\alpha|X|\beta\rangle = \int dx\alpha^*(x)x\beta(x) \tag{3}$$

$$\langle\beta|X|\alpha\rangle = \int dx\beta^*(x)x\alpha(x) \tag{4}$$

It is an easy exercise to show that the above formulas are complex conjugates to each other. Therefore, we have proven that the position operator is Hermitian. The case with the momentum operator is trickier. In position basis, the momentum operator P acts by $-i\hbar\frac{\partial}{\partial x}$. Therefore we can write as follows.

$$\langle\alpha|P|\beta\rangle = \int dx\alpha^*(x)\left(-i\hbar\frac{\partial}{\partial x}\beta(x)\right) \tag{5}$$

$$\langle\beta|P|\alpha\rangle = \int dx\beta^*(x)\left(-i\hbar\frac{\partial}{\partial x}\alpha(x)\right) \tag{6}$$

At first glance, they don't look like complex conjugates to each other. However, the trick is to use integration by parts.

$$
\begin{aligned}
\langle\beta|P|\alpha\rangle &= \int_{-\infty}^{\infty} dx\beta^*(x)\left(-i\hbar\frac{\partial}{\partial x}\alpha(x)\right) \\
&= \int_{-\infty}^{\infty} dx\left(i\hbar\frac{\partial}{\partial x}\beta^*(x)\right)\alpha(x) - (i\hbar\beta^*(\infty)\alpha(\infty) - i\hbar\beta^*(-\infty)\alpha(-\infty))
\end{aligned}
$$

However, a generic wave function should go to zero at infinities. Otherwise, the particle associated with this wave function would spread out to infinity. If your wave function extends to infinity, you will have non-zero probability of finding you at far away galaxies. Therefore, the last two terms on the above expression must be zero. Therefore, we get:

$$
\langle\beta|P|\alpha\rangle = \int_{-\infty}^{\infty} dx\beta^*(x)\left(-i\hbar\frac{\partial}{\partial x}\alpha(x)\right) = \int_{-\infty}^{\infty} dx\alpha(x)\left(i\hbar\frac{\partial}{\partial x}\beta^*(x)\right)
$$

Now, it is a simple exercise that the above expression is conjugate to $\langle\alpha|P|\beta\rangle$. This completes the proof.

Problem 1. Show that $\frac{\partial}{\partial x}$ is anti-Hermitian, while $\frac{\partial^2}{\partial x^2}$ is Hermitian.

Problem 2. The energy operator is called the "Hamiltonian operator" in quantum mechanics and denoted by H. In other words,

$$
H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{7}
$$

Using the result of Problem 1, show that the Hamiltonian operator given by the above formula is Hermitian.

Summary

- To prove $\langle\beta|P|\alpha\rangle = \langle\alpha|P|\beta\rangle^*$, you need to use integration by parts and the fact that the wave function vanishes at $x = \pm\infty$.