Radian

See Fig. 1. How do we calculate "l" which is defined as the length of arc with the radius r and degree θ? It is easy to see that the length should be $\theta/360°$ times the length of the circle with radius r. Therefore, we get the following:

$$l = 2\pi r \frac{\theta}{360°} = r\left(\frac{\pi}{180°}\theta\right) \tag{1}$$

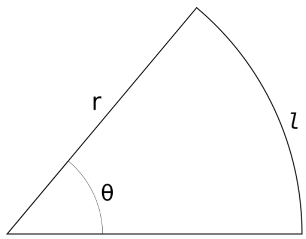

Figure 1: arc with the radius r and degree θ

Now, it may seem that the extra factor $\pi/180°$ is cumbersome. Therefore, let's absorb this factor into the definition of the angle. So, if we define our new θ as our old θ multiplied by $(\pi/180°)$, we may write the above relation as $l = r\theta$. This is the definition of a radian. In other words, we have the following relation:

$$\theta(\text{radian}) = \frac{\pi}{180°}\theta(\text{degree}) \tag{2}$$

For example, the right angle is $90°$. In radians, it's $\frac{\pi}{180°}90° = \frac{\pi}{2}$.

Figure 2: $r\sin\theta \approx r\theta$

What is the advantage of using radians? After all, nobody uses radians in daily life. People always use degrees. However, in mathematics radians are more natural than degrees. Not only do radians have the simple formula for the length of an arc, they can approximate the value of trigonometric functions much more succinctly. See Fig. 2. When θ is very small, you can easily see that "$r \sin\theta$ is approximately "$r\theta$." Therefore, in this case, we have:

$$\sin\theta \approx \theta \tag{3}$$

This fact has an enormous importance in mathematics. Saying it again, it justifies the use of radians and makes it "natural." As a side remark, in our later article "Taylor series," we will prove that the sine function in terms of radian θ is given by the following formula:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots \tag{4}$$

where $n!$ is defined by:

$$n! = n \times (n-1) \times (n-2) \times \cdots 3 \times 2 \times 1 \tag{5}$$

(For example, $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$) The series converges for any θ.

An interesting point is that an angle is "dimensionless" when expressed in radians. Notice that our earlier relation $l = r\theta$ means

$$\theta = \frac{l}{r} \tag{6}$$

The right hand-side is dimensionless, as it is length divided by length. Therefore, θ is indeed dimensionless. Notice that if θ were not dimensionless, we would not be able to directly add each term in (4), because θ, θ^3, θ^5 would all have different dimensions. For example, if l is length, we cannot add l and l^3, because the former has the dimension of length while the latter has the dimension of volume. The fomer must have an extra factor that has dimension of length squared to be added to the latter. Otherwise, the latter must be divided into a factor that has dimension of length square.

Note that if we want to use degrees instead of radians for the formula (4), the same formula becomes much more complicated, precisely from the reason just mentioned. From (2), you can easily see that it is given by:

$$\sin\theta = \frac{\pi}{180^\circ}\theta - \frac{1}{3!}\left(\frac{\pi}{180^\circ}\theta\right)^3 + \frac{1}{5!}\left(\frac{\pi}{180^\circ}\theta\right)^5 - \frac{1}{7!}\left(\frac{\pi}{180^\circ}\theta\right)^7 + \frac{1}{9!}\left(\frac{\pi}{180^\circ}\theta\right)^9 - \cdots \tag{7}$$

Therefore, we indeed see that radians are more natural than degrees. Actually, we also have $\tan\theta \approx \theta$ in radians. This is apparent from the arc and the right triangle drawn in Fig. 3. Again, the point is that this simple relation would be complicated if we used degrees.

Finally, I want to advise you that you have to be careful when you calculate trigonometric functions using a calculator. If you type $\sin 30$ into calculator and it doesn't give 0.5, it probably means that your calculator is set in radian mode rather than degree mode, as $\sin 30$ in radian is not

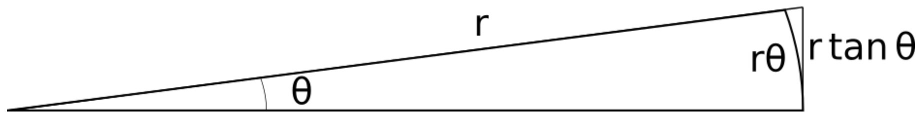

Figure 3: $r\tan\theta \approx r\theta$

0.5. Similarly, if you type $\sin(\pi/6)$ into calculator and it doesn't give 0.5, it probably means that your calculator is set in degree mode rather than radian mode.

Actually, you can also use google to obtain trigonometric functions. If you google "sin 30" it will return the sine for 30 radians. If you google "sin 30 degrees" it will return 0.5, the sine for 30 degrees.

Problem 1. Express the area of the sector drawn in Fig. 1 in terms of r and θ(radian).

Problem 2. What is the period of $\sin x$ when x is given in terms of radians?

Summary

- 180 degrees is π radian.
- The length of the arc l with radius r and radian degree θ is given by

$$l = r\theta$$