Rational numbers and repeating decimal

In our esssay "$\sqrt{2}$ as irrational number," we stated the definition of a rational number: A rational number is a number that can be expressed in the form a/b where a and b are integers. In this article, we will investigate what happens when we express a rational number using decimals. In particular, we will see that a rational number can be always expressed as a type of decimal called "repeating decimal." Also, we will learn how to turn a repeating decimal into a fraction a/b for a and b integers. In other words, we will see that a repeating decimal is always a rational number.

Let's begin. Let's try to divide a by b for two integers. If the answer is not an integer, you will get a decimal. This decimal will either terminate or not terminate. Let me explain what I mean by examples. Here are some examples for terminating decimals.

$$\frac{1}{2} = 0.5, \qquad \frac{737}{500} = 1.474, \qquad \frac{3}{8} = 0.375 \tag{1}$$

Here are some examples for non-terminating decimals.

$$\frac{1}{3} = 0.333\cdots, \qquad \frac{17}{110} = 0.1545454\cdots, \qquad \frac{11}{7} = 1.571428571428\cdots \tag{2}$$

In these examples of non-terminating decimals, you might have noticed a pattern. The decimals repeat. In the first example, the number 3 repeats. In the second example, the numbers 54 repeats. In the third example, the numbers 571428 repeat. Why is this so? Let's figure this out by checking what actually happens when we divide the numbers using long division. Let's check the third example.

$$
\begin{array}{r}
1.57142857 \\
7\overline{)11.00000000} \\
\underline{7} \\
4.0 \\
\underline{3.5} \\
50 \\
\underline{49} \\
10 \\
\underline{7} \\
30 \\
\underline{28} \\
20 \\
\underline{14} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50 \\
\underline{49} \\
1
\end{array}
$$

Now, it is clear why the digits have to repeat. When we divided 11 by 7, the remainder was 4. Then, we add 0 to make it 4.0, and divide again to get the quotient 5 and the remainder 5, to which we add 0 and divide again to get the quotient 7 and the remainder 1 and so on. So, the remainder goes like 4, 5, 1, 3, 2, 6 and 4. Once we hit the remainder 4 again, the pattern has to repeat. If we add 0 to make it 40, divide it again by 7, we get the same quotient and the same remainder as the earlier ones. Then, the pattern has to repeat necessarily, because the same subsequent remainders have to follow. Notice that, when we divide by 7, it is not possible for the remainders not to repeat, as there are only 6 possible remainders, namely, 1 to 6. Eventually, you will necessarily come back to the same remainder again, because there aren't endless possible remainders; you will come back to the original remainder after dividing at most 6 more times.

This explains why any rational number a/b can be expressed as a repeating decimal. If you perform long division, there are at most only $b-1$ possible remainders. Therefore, after dividing at most $b-1$ times, you will come back to one of the earlier remainders.

Notice also that a terminating decimal is a repeating decimal in a broad sense, as we can regard a terminating decimal as having repeating 0 at the end of the decimal.

Given that all rational numbers are repeating decimals, it is right to suspect that any repeating decimal is a rational number. But, we are not sure, before we prove this. After all, there could be a repeating decimal which cannot be represented as a fraction of form a/b. However, as we will see, there is no such repeating decimal. We can prove this, by explicitly finding such a fraction for *any* repeating decimal. Let's do this for a repeating decimal, as the generalization is easy.

To this end, let's first introduce a notation to represent a repeating decimal. The notation differs from countries to countries, but in South Korea, we use dots to denote the repeating part of the decimal. For example,

$$1.571289128912891289\cdots = 1.57\dot{1}28\dot{9} \tag{3}$$

To turn this into a fraction, we can think this as

$$1.57\dot{1}28\dot{9} = \frac{157.\dot{1}28\dot{9}}{100} \tag{4}$$

Thus, if we turn $0.\dot{1}28\dot{9}$ into a fraction, we are done. To this end, notice

$$0.\dot{1}28\dot{9} = 1289 \times 0.\dot{0}00\dot{1} \tag{5}$$

Thus, if we turn $0.\dot{0}00\dot{1}$ into a fraction, we are done. However, it is not hard to check

$$0.\dot{0}00\dot{1} = \frac{1}{9999} \tag{6}$$

(Please check!). Thus,

$$0.\dot{1}28\dot{9} = \frac{1289}{9999} \tag{7}$$

$$1.57\dot{1}28\dot{9} = \frac{157.\dot{1}28\dot{9}}{100} = \frac{157 + \frac{1289}{9999}}{100} = \frac{1571132}{999900} \tag{8}$$

We see that turning a repeating decimal is a little bit involved, but it's not that hard if we do it step by step. Actually, there is a simple quick formula to turn a repeating decimal into a fraction, so that you do not need to take all these steps all again every time you turn a repeating decimal to a fraction. In South Korea, we learn this short-cut formula in middle school. Certainly, it is great to learn how to turn a repeating decimal into a fraction. However, it is very unfortunate that Korean students are forced to memorize this formula for exam without having to understand how this formula is derived. Of course, you may think that a South Korean student doesn't need to memorize this formula, if he/she can find the answer without it, but he/she has to, because he/she has to solve as many problems as possible in a short time. No wonder that South Korean students are ranked one of the best in their math exams, but their interest in math is ranked one of the lowest in OECD member countries. What is important is not the ability to quickly convert a repeating decimal into a fraction, but the ability to derive such a formula by using logic. Teaching mathematics should be about fostering logical thinking. I am sure that an average South Korean middle school student can convert a very complicated repeating decimal like our example, into a fraction faster than 99% of mathematicians including the Fields Medalists, who have never learned or memorized such a formula, but are able to derive such a formula very easily nonetheless. And, of course, the 1% of mathematicians, who can solve this problem faster than an average South Korean middle school student, must be South Korean mathematicians.[1]

Problem 1. The formula we learn in South Korea is the following. If the repeating part of decimal has n digits, and the non-repeating part of decimal has m digits, you write 9 n times, and then you put 0, m times. This becomes the deniminator. For example, $1.57\dot{1}28\dot{9}$ has 4 digits repeating part (1289), and 2 digits non-repeating part (57). Thus, the denominator is 999900. Then, you write out the whole digit, including the repeating part, and then subtract the non-repeating part of the digits. In our case, this is $1571289 - 157$. This is the numerator. Thus, the answer is

$$1.57\dot{1}28\dot{9} = \frac{1571289 - 157}{999900} \tag{9}$$

Explain why this formula must work.

[1] Of course, I am joking here. I am not 100% sure whether there are any other countries that make their middle or high school students memorize this formula.