Recombination

As I explained in an earlier article, recombination is the combination of nucleus and free electron (i.e., ionized electron) to form an atom during our early Universe. Let's calculate the fraction of free electrons in terms of the temperature of our early Universe.

We will only consider the recombination of hydrogen atom in this article, i.e., in case that the nucleus to be combined is proton. Then, we have

$$p + e^- \leftrightarrow H + \gamma \tag{1}$$

Now, we can directly use the result of our earlier article "chemical equilibrium." We need to use

$$\mu_p + \mu_e = \mu_H \tag{2}$$

as the chemical potential of photon is zero.

However, we need to be careful because we have to multiply the number density by the internal degree of freedom g. Considering this, at equilibrium, we have

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{m_H}{m_e m_p} \frac{2\pi}{T} \right)^{3/2} e^{B/T} \tag{3}$$

where B is the binding energy of hydrogen i.e., the energy released when a free electron is captured and falls into $n = 1$ state. It is given by

$$B_H = \frac{\text{Ry}}{1^2} = m_p c^2 + m_e c^2 - m_H c^2 = 13.6\text{eV} \tag{4}$$

where Ry is the Rydberg unit of energy introduced in "Hydrogen atom." g_e is 2, as electron has spin up and spin down state, and g_p is also 2, as it has spin 1/2 just like an electron. So, we see that there are total of $2 \times 2 = 4$ states for hydrogen atom with $n = 1$. Thus, we have $g_H = 4$.

We also know that $n_e = n_p$, because our Universe is neutral. We can also make an approximation $m_H = m_p$ in the parenthesis of right-handside, as the mass of electron is about 1,800 times smaller than the one of proton, but we cannot make this approximation for B i.e., (4). Therefore, we have

$$\frac{n_e^2}{n_H} = \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-B_H/T} \tag{5}$$

So, if we define the ionization fraction X by

$$X \equiv \frac{n_e}{n_e + n_H} \tag{6}$$

(5) can be re-expressed as

$$\frac{X^2}{1-X} = \frac{n_e^2}{n_H(n_p+n_H)} = \frac{1}{n_p+n_H}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-B_H/T} \tag{7}$$

Therefore, you see that there are less and less fractions of free electrons as T drops. Naively, you may think that this happens when T is in the order of B_H i.e., 13.6 eV=158,000 K. However, this is not so. As there are much more photons than protons or hydrogen atoms, even at a significantly lower temperature than 158,000 K, there are sufficiently many energetic photons to ionize the hydrogen atom.

So, let's calculate the temperature that the ioniziation rate significanlty drops below 1. Earlier, we have seen that the helium mass fraction of baryon is about $Y_p = 0.25$. Therefore, $n_p + n_H$ is about $75\% (= 1 - Y_p)$ of the baryon density n_B. Also, cosmologists obtained that baryon to photon ratio now is about

$$\eta \equiv \frac{n_B}{n_\gamma} \approx 6 \times 10^{-10} \tag{8}$$

and we know that this value was hardly different at the recombination, because baryons and anti-baryons were already annhilated long before the recombination that only baryons were left behind and no more siginificant amount of photons were being produced through baryon-anti-baryon annhilations.

From our earlier article on the number density of relativistic gas, we have

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3 \tag{9}$$

Plugging this into

$$n_p + n_H = (1 - Y_p) n_B = (1 - Y_p)\eta n_\gamma \tag{10}$$

then into (7), we obtain the following "Saha equation."

$$\frac{X^2}{1-X} = \frac{1}{(1-Y_p)\eta}\left(\frac{\sqrt{\pi}}{4\sqrt{2}\zeta(3)}\right)\left(\frac{m_e}{T}\right)^{3/2} e^{-B/T} \tag{11}$$

Notice that, given a value of T, we can obtain the right-hand side. Then, the expression is in the form of a quadratic equation of X. Therefore, we can obtain X as a function of T. See a graph of X in terms of T in Fig. 1. Recombination occured between 3,000 K∼4,000 K.

I will upload the figure later.

However, for an accurate calculation, cosmologists consider the recombination of helium, and the fact that a perfect equilibrium is not maintained, as many energetic photons are released by recombination of the nuclei and the electrons, which make the distribution of photon deviate from the Maxwell-Boltzmann one. Notice that such photons are at the energy level which are crucial to recombination making such a consideration more important; they can ionize the first neutral hydrogen atom they meet.

Summary

- Recombination is a combination of nuclei and electrons to form neutral atoms during our early Universe.
- Recombination occured at a temperature much lower than the ionization energy of hydrogen because of very low baryon to photon ratio. As there are much more photons than protons or hydrogen atoms, even at a significantly lower temperature than the ionization energy, there are sufficiently many energetic photons to ionize the hydrogen atom.
- Saha equation gives the ionization rate of electron as a function of temperature.