Reduced mass

When we derived Kepler's third law in an earlier article, we assumed that the Sun doesn't move at all and only the planets orbit around the Sun. However, strictly speaking, this is not correct. According to Newton's third law, the Sun is attracted by planets by the same amount of forces as the Sun attracts the planets. Therefore, the Sun does move, even though it does by a tiny amount as the Sun is much heavier than the planets.

In this article, we will consider such a case in which the movement of the heaviest object cannot be neglected. Furthermore, to simplify the problem, we will consider the case in which only two objects are concerned, as we exactly did so in our earlier article on Kepler's third law by neglecting the gravitational attraction between planets. (We only plugged in Sun's gravitational force when we equated it with centripetal force.)

From Newton's third law, we know that the total momentum is conserved. Furthermore, if you remember our discussion in "Elastic collision in 1-dimension in center of mass frame," you will know that the center of mass doesn't move in the center of mass ("C.M.") frame. Therefore, if we consider a system with the Earth and the Moon only, the Moon doesn't rotate around the Earth as its center, but around the center of mass of the Earth and the moon. Similarly, the Earth doesn't stay idly, but rotates around the center of mass of the two bodies. See Fig. 1. In reality, the center of mass of Earth-Moon system is inside the Earth.

Now, let's write out some equations. Let's say the object A has mass m_A, and the object B, m_B, and they are apart by the distance r as in the figure. Then, if the origin of the coordinate system is located at the center of mass, and if we denote $\vec{r}$ the relative position of B with respect to A, B's location $\vec{r}_B$ is given by

$$\vec{r}_B = \frac{m_A}{m_A + m_B}\vec{r} \tag{1}$$

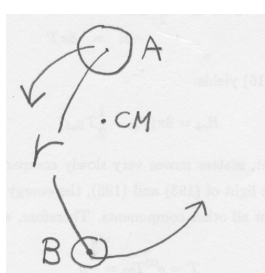

Figure 1: Two objects A and B orbiting around the center of mass

Given this, if the force between the two bodies is given by $\vec{F}$, we have:

$$m_B \ddot{\vec{r}}_B = \vec{F} \tag{2}$$

$$\frac{m_A m_B}{m_A + m_B} \ddot{\vec{r}} = \vec{F} \tag{3}$$

So, the structure of the equation is same as if the mass of lighter object were $m_A m_B/(m_A + m_B)$, and the heavier object not moving at all, provided that we plug in the same F. We call this mass "reduced mass" as it is smaller than both m_A and m_B. Notice that as expected, the reduced mass is equal to the mass of the lighter object, if the mass of the heavier object is infinite. (**Problem 1.** Show this.)

Problem 2. What is the orbiting period of two objects with the same mass m rotating each other along circular orbit, if the distance between them is $2r$?

Problem 3. What is the oscillation period of a spring with spring constant k, if on object with mass m is attached at one end, and another object with mass $2m$ is attached at the other end?

Summary

- If two objects with mass m_A and m_B interact each other through force $\vec{F}$, we can write

  $$\mu \ddot{\vec{r}} = \vec{F}$$

  where $\vec{r}$ is the distance between the two objects, and μ is the reduced mass given by

  $$\mu = \frac{m_A m_B}{m_A + m_B}$$

- So, the structure of the equation is same as if the mass of lighter object were $m_A m_B/(m_A + m_B)$, and the heavier object not moving at all, provided that we plug in the same F.

- $\mu < m_A, m_B$ and when m_B is going to the limit infinity, $\mu = m_A$, as expected.