Reflection and transmission of travelling wave

In this article, we will quantitatively describe reflection and transmission of travelling wave when it hits upon a different medium. Then, you will be asked to check the assertion stated about reflected pulse in our earlier article "Interference from thin films."

See Fig.1. in that article. A pulse is coming from the left. After it hits the interface, a transmitted pulse goes to the right and a reflected pulse goes to the left. Interface is the point where the two different media meet. We will call the speed of the wave on the medium situated on the left of the interface by v_L and the one on the right by v_R. Then Fig.1.(a) corresponds to the case $v_L < v_R$ and Fig.1.(b) corresponds to the case $v_L > v_R$. We will calculate the reflection amplitude and the transmission amplitude in a general case without considering the both cases separately. Then, plugging in the appropriate values for v_L and v_R will yield the results for the both cases.

In that article, it was a pulse that we considered. In this article, we will consider sine waves. This is justified since we know that a pulse is a sum of sine waves from our earlier articles on Fourier series; after obtaining the reflection wave and the transmission wave in the case when the incident wave is a sine wave, we can sum up the corresponding sine-like reflection waves and the sine-like transmission waves to obtain the reflected pulse and the transmitted pulse.

Anyhow, let's say that the interface is at $x = 0$. Let's also express the incident wave as follows:

$$A_c \cos(k_L x - \omega t) + A_s \sin(k_L x - \omega t) \tag{1}$$

This makes sense since the incident wave is moving to the right. We also naturally have $k_L = \omega/v_L$. The above formula is valid for $x \leq 0$, as the incident wave is currently on the left of the interface.

Now, as the transmitted wave is moving to the right, we can similarly express it as follows:

$$F_c \cos(k_R x - \omega t) + F_s \sin(k_R x - \omega t) \tag{2}$$

where $k_R = \omega/v_R$. The above formula is valid for $x \geq 0$ as the transmitted wave is on the right of the interface.

Now, let's consider the reflected wave. The reflected wave moves with the same speed with the incident wave as they are in the same medium but in the opposite direction. Therefore, we have:

$$B_c \cos(-k_L x - \omega t) + B_s \sin(-k_L x - \omega t) \tag{3}$$

This equation is valid for $x \leq 0$.

Summarizing, when $x \leq 0$ the wave is given by sum of (1) and (3) and when $x \geq 0$ the wave is given by (2). In other words, if we call the wave function by $\psi(x,t)$, for $x \leq 0$, we have:

$$\psi(x,t) = A_c \cos(k_L x - \omega t) + A_s \sin(k_L x - \omega t) + B_c \cos(-k_L x - \omega t) + B_s \sin(-k_L x - \omega t) \quad (4)$$

and for $x \geq 0$, we have:

$$\psi(x,t) = F_c \cos(k_R x - \omega t) + F_s \sin(k_R x - \omega t) \quad (5)$$

Now comes the crucial point. The wave function cannot have two values at the same point. Otherwise, it would mean that the rope is torn. Since (4) and (5) are continuous functions, the only part we have to be careful about the continuity of the wave function is at $x = 0$. Plugging this value to (4) and (5) and equating them, we get:

$$A_c + B_c = F_c, \qquad A_s + B_s = F_s \quad (6)$$

Now comes another crucial point. The first derivative of the wave function must be continuous. Otherwise, it would mean that the rope is not smooth, but angled at the discontinuous point. This implies $\dfrac{\partial \psi(x,t)}{\partial x}$ must be continuous. Again, we need to consider when $x = 0$. Plugging (4) and (5), we obtain:

$$k_L A_c - k_L B_c = k_R F_c, \qquad k_L A_s - k_L B_s = k_R F_s \quad (7)$$

All we are left to do is solve (6) and (7). The incident wave is given. Therefore, we know A_c and A_s. The unknowns are B_c, B_s, F_c and F_s. If you solve them, we get (**Problem 1.** Check this!)

$$\frac{B_c}{A_c} = \frac{B_s}{A_s} = \frac{k_L - k_R}{k_L + k_R} \quad (8)$$

$$\frac{F_c}{A_c} = \frac{F_s}{A_s} = \frac{2k_L}{k_L + k_R} \quad (9)$$

In other words, (8) is the ratio of the amplitude of the reflected wave to the one of the incident wave, and (9) is the ratio of the amplitude of the transmitted wave to the one of the incident wave. Notice also that the quantity in (8) is always between -1 and 1. This makes sense since a reflected wave can never be bigger than the incident wave.

Let me conclude this article with some remarks. Even though we consider the wave on a rope in this article, everything in this article can be carried into any other cases as long as they deal with the reflection and the transmission of 1-dimensional waves. For example, if you send a beam of light from the air to the water in a direction perpendicular to the surface of the water, (8) and (9) hold. Nevertheless, more analysis is needed when the beam of light is hit upon the surface of water askew.

Problem 1. What happens when a wave is hit upon the same medium? Show that there is no reflected wave and every wave is transmitted as expected since it can be regarded as not hitting new medium at all. (Hint[1])

[1] In such a case $k_L = k_R$

Problem 2. Check the assertion made about the reflected pulse in our earlier article "Interference from thin films." (Hint[2])

Problem 3. Remember that we have alternatively expressed the sine and cosine functions by exponential functions in Problem 5. in our earlier article "Differential equations." Similarly, we can revisit all our calculations in this article using exponential functions and this makes the calculation simpler. For example, (1), (2) and (3) can be re-expressed as:

$$A\exp(i(k_L x - \omega t)), \quad F\exp i(k_R x - \omega t), \quad B\exp(i(-k_L x - \omega t)) \tag{10}$$

Using these relations, re-do the calculation done in this article and re-obtain (8) and (9).

Summary

- When a travelling wave enter a different medium (i.e. the one that has a different wave speed), part or all of the wave is reflected and part of all of the wave is transmitted.
- The reflection amplitude and the transmission amplitude can be obtained from the condition that the wave function and the first derivative of wave function are continuous.

[2]Think about the sign of (8)