The number density, the energy density and the entropy density of relativistic gas

In this article, we will derive expressions for the number density, the energy density, and the entropy density of relativistic gas. A gas is a collection of particles, and a relativistic gas is a gas that has a temperature much higher than the rest mass of its constituent particles. (In natural unit $k = c = 1$, the temperature has the same dimension as the energy, which has the same dimension as the mass.) Therefore, most particles in the relativistic gas move at a speed that is very close to the speed of light. Therefore, we will ignore the rest mass in our calculation for relativistic gas. We further assume that the chemical potential is also negligible, as this is our case of interest later when we use the formulas we derive here.

Recall that the number of states is given by

$$g\frac{d^3pd^3q}{h^3} \tag{1}$$

where g is the internal degrees of freedom. In case of photon, $g = 2$ because there are two polarizations. In case of an electron, we have also $g = 2$, because there are two states, namely, spin up state and spin down state. Similarly, a neutrino has $g = 2$.

Thus, the number of states per volume is given by

$$g\frac{d^3p}{h^3} = g\frac{4\pi^2dp}{(2\pi)^3\hbar^3} = \frac{gp^2dp}{2\pi^2} \tag{2}$$

where in the last step we used the natural unit $\hbar = 1$.

Then, considering the Bose-Einstein distribution and the Fermi-Dirac distribution, the number density is given by

$$n = \int \frac{gp^2dp}{2\pi^2}\frac{1}{e^{p/T} \pm 1} = \frac{g}{2\pi^2}T^3\int_0^\infty \frac{x^2\,dx}{e^x \pm 1} \tag{3}$$

where we used the fact that the energy of a relativistic particle with p is $E = pc = p$ in the natural unit (i.e. $c = 1$).

It turns out that

$$\int_0^\infty \frac{x^3dx}{e^x - 1} = 2\zeta(3), \qquad \int_0^\infty \frac{x^3dx}{e^x - 1} = \frac{3}{4}(2\zeta(3)) \tag{4}$$

where $\zeta(s)$ is the Riemann-zeta function defined by

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \tag{5}$$

Therefore, the number density of a relativistic gas is given by

$$n = g\frac{\zeta(3)}{\pi^2}T^3 \quad \text{for bosons} \tag{6}$$

$$n = \frac{3}{4}g\frac{\zeta(3)}{\pi^2}T^3 \quad \text{for fermions} \tag{7}$$

Similarly, the energy density of a relativistic gas is given by

$$\rho = \int \frac{gp^2dp}{2\pi^2}\frac{p}{e^{p/T} \pm 1} = \frac{g}{2\pi^2}T^4\int_0^\infty \frac{x^3\,dx}{e^x \pm 1} \tag{8}$$

It turns out that

$$\int_0^\infty \frac{x^3dx}{e^x - 1} = \frac{\pi^4}{15}, \qquad \int_0^\infty \frac{x^3dx}{e^x - 1} = \frac{7}{8}\frac{\pi^4}{15} \tag{9}$$

Thus, we have

$$\rho = g\frac{\pi^2}{30}T^4 \quad \text{for bosons} \tag{10}$$

$$\rho = \frac{7}{8}g\frac{\pi^2}{30}T^4 \quad \text{for fermions} \tag{11}$$

We know that the pressure is given by $P = \rho/3$ for relativistic gas, so we can easily calculate the pressure from the above formula.

Now, let's calculate the entropy density $s \equiv dS/dV$. Using $\rho = dU/dV$, we have

$$TdS = dU + PdV \tag{12}$$

$$dS = \frac{\rho dV + PdV}{T} \tag{13}$$

$$s = \frac{\rho + P}{T} \tag{14}$$

Thus, for a relativistic gas, we have

$$s = \frac{4\rho}{3T} \tag{15}$$

which implies

$$s = g\frac{2\pi^2}{45}T^3 \quad \text{for bosons} \tag{16}$$

$$s = \frac{7}{8}g\frac{2\pi^2}{45}T^3 \quad \text{for fermions} \tag{17}$$

Problem 1. Show that, if we don't assume that a gas is relativistic, (8) becomes

$$\rho = \frac{g}{2\pi^2}T^4\int_0^\infty \frac{x^2\sqrt{x^2 + (m/T)^2}dx}{e^{\sqrt{x^2+(m/T)^2}} \pm 1} \tag{18}$$

Then, show that the above expression reduces to (8) in the relativistic limit (i.e., $T \gg m$).

Summary

- A gas is a collection of particles, and a relativistic gas is a gas that has a temperature much higher than the rest mass of its constituent particles.