Revisiting Gauss's law and the derivation of Coulomb's law

Gauss's law is stated as follows:

$$\oint E \cdot dA = \frac{q}{\epsilon_0} \tag{1}$$

where the surface integration is a closed Gaussian surface and q on the right hand side is the net charge enclosed in the Gaussian surface and ϵ_0 is the proportionality constant that relates the left-hand side and the right-hand side. Except for ϵ_0, all this should be familiar with you if you read my earlier article "Gauss's law" and "Flux." Now, let's derive Coulomb's law. If the charge q is located at the center of spherical Gaussian surface with radius r, we have:

$$E \cdot 4\pi r^2 = \frac{q}{\epsilon_0} \tag{2}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \tag{3}$$

if we let $k \equiv 1/(4\pi\epsilon_0)$, the above equation can be re-written as:

$$E = \frac{kq}{r^2} \tag{4}$$

This implies exactly Coulomb's law, since in the presence of another charge Q at the location of the electric field, we have:

$$F = QE = \frac{kQq}{r^2} \tag{5}$$

Problem 1. Consider a hollow sphere with radius r and charge q distributed homogeneously on the surface. What is the electric field inside the sphere and outside the sphere?

Problem 2. See Fig.1. We have a rod with length L on which the charge $+q$ is uniformly distributed. Also, the point P is the distance a far from one of the tip of the rod, and the distance $a + L$ far from the other tip. Notice also that the charge per unit length is given by q/L. Given this, obtain the electric field given at the point P by integration. (Hint[1]) Show also that the electric field is approximately given by $E = kq/a^2$ when a is much larger than L. This shows that the rod can be seen as a single point with charge q in this limit.

Problem 3. See Fig.2. We have a ring with radius R on which the charge $+q$ is uniformly distributed. Also, the point P is the distance z above from the center of the ring. Notice

[1] Consider how much electric field is contributed from the charge in the dx segment drawn in the figure. This will give you dE in terms of dx and x. Then integrate this quantity from a to $a + L$.

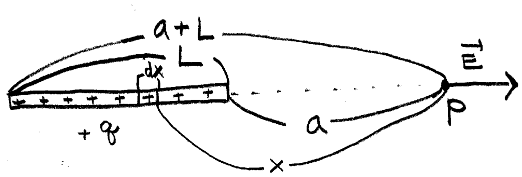

Figure 1: a uniformly charged rod

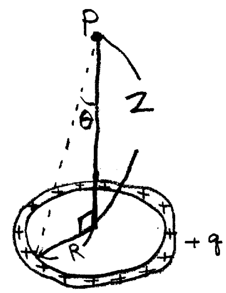

Figure 2: a uniformly charged ring

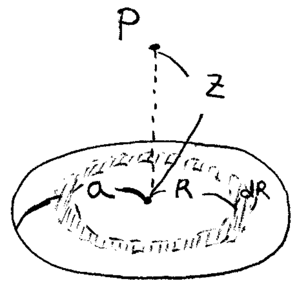

Figure 3: a uniformly charged disk

also that this point is $\sqrt{z^2 + R^2}$ meter far from where the actual charges are. Given this, find the electric field at the point P. (Hint[2])

Problem 4. See Fig.3. We have a disk with radius a on which a uniform charge density σ is distributed. (The total charge on the disk is given by the area times the charge density which is $\sigma\pi a^2$). Also, the point P is the distance z above from the center of the disk. Given this, calculate the electric field at the point P. (Hint. In the figure, you see the shaded region with the area $2\pi R dR$. Calculate what contributions the charge in this region has on the electric field and integrate the result from R being 0 to a. You will need to use the result of Problem 3.) Show also that the limit when a approaches infinity, the electric field at P converges to a certain constant value. What is this value? In "Application of Gauss' law" we will obtain it using another method without using a complicated calculation as in this article.

Summary

- Coulomb's law can be derived from Gauss's law.

[2]Notice that the electric field is in z direction, as the electric field along x and y direction cancel one another because of the symmetry. It suggests that all we need to do is just calculating the z component of each electric field contributed from each infinitesimal segment of the ring and integrate them. Notice that each electric field makes an angle θ with the z axis, so its z component is $\cos\theta$ multiplied by it.