Revisiting Fourier transformations

As promised, we will derive the Fourier transformation all again in this article. Let me repeat the problem. We want to find a_n and b_n in the following Fourier series.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nt) + b_n \sin(nt)\right] \tag{1}$$

Now, let's regard $f(t)$ as a vector, which can be expressed as a linear combination of the following basis vectors.

$$1 = |1\rangle, \quad \cos(nt) = |c_n\rangle, \quad \sin(nt) = |s_n\rangle \tag{2}$$

Then, (1) can be re-expressed as

$$|f\rangle = \frac{a_0}{2}|1\rangle + \sum_{n=1}^{\infty} (a_n|c_n\rangle + b_n|s_n\rangle) \tag{3}$$

Now, I will define a dot product of two vectors as follows:

$$\langle g|h\rangle = \int_0^{2\pi} g(t)h(t)dt \tag{4}$$

It obviously satisfies $\langle g|h\rangle = \langle h|g\rangle$. Given this, we can check that the basis vectors are orthogonal to one another:

$$\langle 1|1\rangle = 2\pi, \quad \langle c_n|c_m\rangle = \langle s_n|s_m\rangle = \pi\delta_{nm}, \quad \langle 1|c_n\rangle = \langle 1|s_n\rangle = \langle c_n|s_m\rangle = 0 \tag{5}$$

Using this orthogonality, let's find a_0 by dot-producting $|f\rangle$ with $|1\rangle$.

$$\langle 1|f\rangle = \frac{a_0}{2}\langle 1|1\rangle + \sum_{n=1}^{\infty}(a_n\langle 1|c_n\rangle + b_n\langle 1|s_n\rangle) = \pi a_0. \tag{6}$$

Thus, $a_0 = \frac{1}{\pi}\langle 1|f\rangle$. Let's find a_m by a similar method.

$$\langle c_m|f\rangle = \frac{a_0}{2}\langle c_m|1\rangle + \sum_{n=1}^{\infty}(a_n\langle c_m|c_n\rangle + b_n\langle c_m|s_n\rangle) = \pi \sum_{n=1}^{\infty} \delta_{nm} a_n = \pi a_m \tag{7}$$

Thus, $a_m = \frac{1}{\pi}\langle c_m|f\rangle$.

Problem 1. Show similarly, $\langle s_m|f\rangle = \pi b_m$. Thus, $b_m = \frac{1}{\pi}\langle s_m|f\rangle$

At this point, you may wonder what are the reasons we can regard functions such as $f(t)$ as a vector. You will find them in "A short introduction to quantum mechanics II: why is a wave function a vector?" There, I showed the reasons why a wave function can be regarded as a vector, but $f(t)$, any function of one variable can be regarded as a vector from the same reasons.

Summary

- if we define a dot product of two vectors as

$$\langle g|h\rangle = \int_0^{2\pi} g(t)h(t)dt$$

each Fourier-mode in Fourier series forms orthogonal basis.