# Revisiting the volume of cone

In our earlier article "Cavalieri's principle and the volumes of cones and spheres," we derived that the cone with a base area $A$ and height $h$ has the volume

$$V = \frac{1}{3}Ah \tag{1}$$

In this article, we will derive this formula more algebraically and rigorously, having learned the concept of limit.

See Fig. 1 for a cone with the base area $A = b^2$ and the height $h$. To approximate the volume, we divided the height into $n$ equal heights, where $n$ in Fig. 1 is 5, and made blocks that inscribe the cone. The total volume of the blocks will be approximately equal to the volume of the cone. What we are going to do now is obtaining a formula for the total volume of the blocks when $n = 5$, then generalize this into an arbitrary $n$. When we send the limit $n$ goes to infinity, the total volume of the blocks will be the volume of cone.

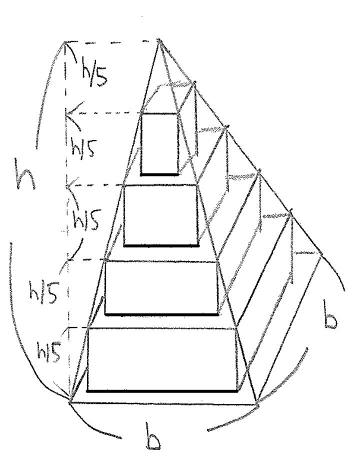


Figure 1: a cone with base area $A = b^2$ and height $h$

Notice that the base area of the smallest block (the one at the top) is given by $(b/5)^2$. The one of the second smallest block is given by $(2b/5)^2$, the one of the third smallest block is given by $(3b/5)^2$, and the one of the fourth smallest block (the one at the bottom) is given by $(4b/5)^2$. Recalling that all these blocks have height $h/5$, we can express the total volume of the blocks by

$$V_5 = \left(\frac{b}{5}\right)^2 \frac{h}{5} + \left(\frac{2b}{5}\right)^2 \frac{h}{5} + \left(\frac{3b}{5}\right)^2 \frac{h}{5} + \left(\frac{4b}{5}\right)^2 \frac{h}{5} \tag{2}$$

$$= \sum_{k=1}^{4} \left(\frac{kb}{5}\right)^2 \frac{h}{5} = \sum_{k=1}^{5-1} \left(\frac{kb}{5}\right)^2 \frac{h}{5} \tag{3}$$

Here 5 in $V_5$ denotes that we divided the cone into five equal heights. More generally, it is easy to see that the above expression becomes

$$V_n = \sum_{k=1}^{n-1} \left(\frac{kb}{n}\right)^2 \frac{h}{n} = \frac{b^2 h}{n^3} \sum_{k=1}^{n-1} k^2 \tag{4}$$

In an earlier article, you proved that

$$\sum_{s=1}^{n} s^2 = \frac{n(n+1)(2n+1)}{6} \tag{5}$$

Thus,

$$V_n = \frac{b^2 h}{n^3} \frac{(n-1)n(2n-1)}{6} \tag{6}$$

Let's take the final step.

**Problem 1.** Show

$$\lim_{n \to \infty} V_n = \frac{b^2 h}{3} \tag{7}$$

Thus, we proved (1).

**Problem 2.** Follow the similar steps as in this article to obtain the area of a triangle with a base $b$ and height $h$.

In this article, you derived the volume of cones hard way. However, once you learn "integration," you won't need to perform this summation, and calculate the volume of cones much easily, as we will see.

## Summary

- We can use a formula for the summation of squares to show that the volume of cones is given by $V = Ah/3$ where $A$ is the base area and $h$ is the height.