Kepler's first and third laws revisited

In this article, by closely following *Analytical Mechanics* by Fowles and Cassiday, we will prove Kepler's first law and third law rigorously. We won't prove Kepler's second law as our earlier treatment of it was already rigorous. To this end, it turns out to be useful to define the following variable.

$$u \equiv \frac{1}{r} \tag{1}$$

In this variable, angular momentum divided by mass of the planet, l is given as follows:

$$l = r^2\dot{\theta} = \frac{\dot{\theta}}{u^2} \tag{2}$$

Given this, by expressing the time derivative in terms of θ derivative, we can obtain $\ddot{r}$ as follows:

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{d\theta}{dt}\frac{du}{d\theta} = -l\frac{du}{d\theta} \tag{3}$$

$$\ddot{r} = -l\frac{d}{dt}\frac{du}{d\theta} = -l\frac{d\theta}{dt}\frac{d}{d\theta}\frac{du}{d\theta} = -l\dot{\theta}\frac{d^2u}{d\theta^2} = -l^2u^2\frac{d^2u}{d\theta^2} \tag{4}$$

Now, remember that we had following in our earlier article,

$$F_{\text{eff}} = m\ddot{r} = \frac{L^2}{mr^3} - \frac{GMm}{r^2} \tag{5}$$

Thus,

$$\begin{aligned} \ddot{r} &= \frac{(L/m)^2}{r^3} - \frac{GM}{r^2} && (6) \\ -l^2u^2\frac{d^2u}{d\theta^2} &= l^2u^3 - GMu^2 && (7) \\ \frac{d^2u}{d\theta^2} + u &= \frac{GM}{l^2} && (8) \end{aligned}$$

The solution to the above differential equation is given by

$$u = B\cos(\theta - \theta_0) + \frac{GM}{l^2} \tag{9}$$

or

$$r = \frac{1}{GM/l^2 + B\cos(\theta - \theta_0)} \tag{10}$$

for a certain B, and θ_0 determined by initial conditions. However, we can set $\theta_0 = 0$ by rotating the polar coordinate system. This corresponds to choosing the closest approach of planet to the Sun as $\theta = 0$. This yields:

$$r = \frac{l^2/(GM)}{1 + (Bl^2/(GM))\cos\theta} \tag{11}$$

First, notice that the orbit closes itself, as the planet comes to the same position after θ changes by 2π. (i.e. $r(\theta) = r(\theta+2\pi)$. Second, further notice that this is precisely an equation for conics, if you remember our earlier discussion in "Conic sections in polar coordinate." Remember, we had

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{12}$$

where e is eccentricity, and a is the semi-major axis of the ellipse in case $-1 < e < 1$. Therefore, we algebraically proved Kepler's first law that the orbits of objects under inverse-square law force are always conics, which Newton had proved geometrically.

Now let's prove Kepler's third law. We know that the area of ellipse is given by πab where b is semi-minor axis. Also, we know the following relation:

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2} \tag{13}$$

which is a constant. Given this, the period (i.e. the time the planet takes to swipe the area of the ellipse) is given by:

$$T = \frac{A}{dA/dt} = \frac{2\pi ab}{l} = \frac{2\pi a^2\sqrt{1-e^2}}{l} \tag{14}$$

which yields:

$$T^2 = \frac{4\pi^2 a^4}{l^2}\frac{a(1-e^2)}{a} \tag{15}$$

Now, comparing (11) and (12), we have:

$$a(1-e^2) = \frac{l^2}{GM} \tag{16}$$

Now plugging this equation to (15), we get:

$$T^2 = \frac{4\pi^2}{GM}a^3 \tag{17}$$

which is Kepler's third law.

Summary

- If you express the motion of a planet in terms of $u \equiv 1/r$, its equation of motion becomes $\dfrac{d^2u}{d\theta^2} + u = \text{const}$. Thus, we can easily solve the equation of motion using sine or cosine function. Expressed this way, we can prove Kepler's first and third laws.