Rolling down on an inclined plane

In an earlier article, we have calculated the speed of an object sliding on an inclined plane. In this article, we will calculate the speed of an object rolling down on an inclined plane without slipping. We will first solve this problem using the conservation of energy. Then, we will solve this problem for the second time using the concepts of force and torque.

For the first approach, we need to prove a theorem; we need to express the kinetic energy of particles in a system in terms of the center of mass velocity and the velocities relative to the center of mass. Let's denote the mass of each particle by m_i and the velocity of each particle by $\vec{v}_i$. Then, the total mass M is given by

$$M = \sum_i m_i \tag{1}$$

and the center of mass velocity is given by

$$\vec{v}_{\mathrm{cm}} = \frac{\sum_i m_i \vec{v}_i}{M} \tag{2}$$

Then, the velocity relative to the center of mass is given by

$$\vec{v}_i' = \vec{v}_i - \vec{v}_{\mathrm{cm}} \tag{3}$$

The total kinetic energy of the particles can be expressed as

$$\begin{aligned}
\frac{1}{2}\sum_i m_i \vec{v}_i \cdot \vec{v}_i &= \frac{1}{2}\sum_i m_i (\vec{v}_{\mathrm{cm}} + \vec{v}_i') \cdot (\vec{v}_{\mathrm{cm}} + \vec{v}_i') \\
&= \frac{1}{2}\sum_i \left(m_i \vec{v}_{\mathrm{cm}}^2 + 2 m_i \vec{v}_i' \cdot \vec{v}_{\mathrm{cm}} + m_i \vec{v}_i'^2\right) \\
&= \frac{1}{2}\left(\sum_i m_i\right)\vec{v}_{\mathrm{cm}}^2 + \left(\sum_i m_i \vec{v}_i'\right) \cdot \vec{v}_{\mathrm{cm}} + \sum_i \frac{1}{2} m_i \vec{v}_i'^2 \qquad (4) \\
&= \frac{1}{2} M \vec{v}_{\mathrm{cm}}^2 + \sum_i \frac{1}{2} m_i \vec{v}_i'^2 \qquad (5)
\end{aligned}$$

where from (4) to (5) we used (1) and

$$\sum_i m_i \vec{v}_i' = 0 \tag{6}$$

In other words, if we notice that $\vec{v}_i'$s are the velocities in the center of mass frame, it is natural that the total momentum in the center of mass frame is zero.

Problem 1. Prove (6) mathematically! (Hint[1])

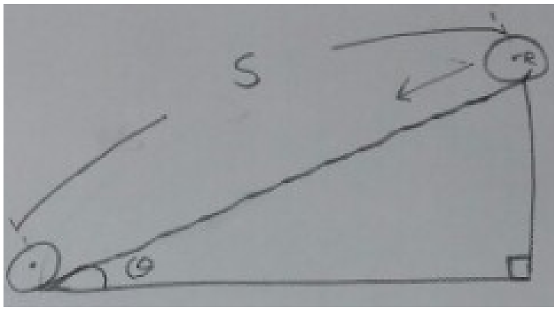

Figure 1: Rolling down on an inclined plane

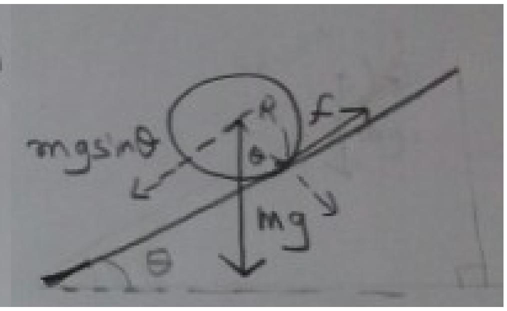

Figure 2: Force diagram

Problem 2. Show that among all the inertial reference frames, the total kinetic energy of particles is smallest in the center of mass frame. Now, we are ready to solve the problem. See Fig. 1. What is the speed of the ball after it rolls down s meter along the inclined plane with slope θ? We will assume that the center of mass of the ball is at the center of the ball, and the mass of the ball is M, the moment of inertia I and the radius R.

We know that the total kinetic energy of the ball after rolling down s meter must be $Mgs\sin\theta$ from the conservation of energy. On the other hand, from the result of the last article, we know

$$\sum_i \frac{1}{2}m_i\vec{v}_i'^2 = \frac{1}{2}I\omega^2 \tag{7}$$

Thus, from (5) the total kinetic energy of the ball with speed v and angular velocity ω is given by

$$\frac{1}{2}Mv^2 + \sum_i \frac{1}{2}m_i\vec{v}_i'^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \tag{8}$$

We also know that the following condition must be satisfied if the ball is not slipping:

$$v = R\omega \tag{9}$$

Thus, we have

$$Mgs\sin\theta = \frac{1}{2}Mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2 \tag{10}$$

Thus, we conclude

$$v = \sqrt{\frac{2gs\sin\theta}{1+\frac{I}{MR^2}}} \tag{11}$$

Now, the second approach. We will approach this problem by analyzing the forces acting on the ball. Now, there are three forces acting on the ball: gravity, normal force, and friction. See Fig. 2. In this case, we have to include the friction, because the ball would never rotate without it. And, the ball has to rotate because we are assuming that the ball rolls down without slipping. Given this, as before, we can decompose gravity to the force parallel to the inclined plane, and to the force normal to the inclined plane. The latter is canceled by the

[1] Use (2) and (3).

normal force. Thus, if we denote the friction force by f, the total force acting on the ball is given by $Mg\sin\theta - f$. Therefore, the acceleration of the ball is given by

$$a = \frac{mg\sin\theta - f}{m} = g\sin\theta - \frac{f}{m} \tag{12}$$

Now, we need to calculate the torque exerted on the ball to calculate the angular acceleration. However, we need to be careful, because the axis of rotation is moving as well! The angular momentum is usually defined by $L = \sum_i \vec{r}_i \times m_i\vec{v}_i$, but now we need to consider

$$L' = \sum_i \vec{r}_i' \times m_i\vec{v}_i' \tag{13}$$

where $\vec{r}_i'$ is the relative position with respect to the axis of rotation, and $\vec{v}_i' = \dot{\vec{r}}_i'$. In other words, if we denote the absolute (i.e. usual) position by $\vec{r}_i$, then we have

$$\vec{r}_i = \vec{r}_i' + \vec{r}_0 \tag{14}$$

where $\vec{r}_0$ is the position of the axis of rotation. Given this, we can write

$$\frac{dL'}{dt} = I\alpha \tag{15}$$

as I and α are determined from the relative position to the axis of rotation, not from the absolute position.

OK. From (13), we have

$$\begin{aligned} \frac{dL'}{dt} &= \sum_i \vec{v}_i' \times m_i\vec{v}_i' + \sum_i r_i' \times m_i\dot{\vec{v}}_i' &(16)\\ &= 0 + \sum_i \vec{r}_i' \times m_i\dot{\vec{v}}_i - \sum_i \vec{r}_i' \times m_i\dot{\vec{v}}_0 &(17)\end{aligned}$$

where from (16) to (17) we used $\vec{v}_i = \vec{v}_i' + \vec{v}_0$ which can be deduced from (14). The second term in (17) is equal to $\sum_i \vec{r}_i' \times \vec{F}_{i(\mathrm{ext})}$ which can be easily deduced by taking the similar step to the one in our earlier article "Re-visiting angular momentum conservation in central force." Thus, (17) becomes

$$\frac{dL'}{dt} = \sum_i \vec{r}_i' \times \vec{F}_{i(\mathrm{ext})} - \left(\sum_i m_i\vec{r}_i'\right)\dot{\vec{v}}_0 \tag{18}$$

In our case, the axis of rotation coincides with the center of mass. Thus, we have $\sum_i m_i\vec{r}_i' = 0$. Therefore, the last term can be dropped. Thus, we can write

$$\frac{dL'}{dt} = \sum_i \vec{r}_i' \times \vec{F}_{i(\mathrm{ext})} \tag{19}$$

We know that the left-hand side is given by $I\alpha$. Now, let's calculate the right-hand side. The gravity acts on the center of mass, so it does not contribute to the torque. The normal force is directed toward the axis of rotation, so it does not contribute to the torque. As you can see clearly from the figure, only the friction contributes to the torque. Thus, we can write

$$I\alpha = fR \tag{20}$$

If the ball rolls down without sleeping, we must have

$$\alpha = \frac{a}{R} \tag{21}$$

Thus, plugging (20) and (21) to (12), we get

$$a = \frac{g \sin\theta}{1 + \frac{I}{MR^2}} \tag{22}$$

Now, it's a simple exercise to derive (11) from this acceleration.

Final comment. You see that the last term in (18) can be dropped when $\dot{\vec{v}}_0' = 0$ as well. In other words, when the axis of rotation is moving at a constant velocity or at rest. We will consider such cases in the next two articles.

Summary

- The total kinetic energy of particles in a system is given by

$$\frac{1}{2} M v_{\mathrm{cm}}^2 + \sum_i \frac{1}{2} m_i v_i^2$$

where M is the total mass of the particles, v_{cm} is the speed of center of velocity, and the second term is the kinetic energy of the particles seen in the center of mass frame.

- Thus, if the center of mass of an object with mass M moves with velocity v, and the object rotates around the center of mass as its axis with angular velocity ω, with the moment of inertia being I, the total kinetic energy of the object is given by

$$\frac{1}{2} M m v^2 + \frac{1}{2} I \omega^2$$