Rotation

Suppose a rigid object rotates around an axis situated at the origin as in Fig.1, and you want to describe the motion mathematically. What variable is useful? You could perhaps say something like "the object rotates 60 degree in 1 second." Therefore, we see that angle is the most natural variable that can describe rotation. Angle θ is denoted in the figure. We have picked up an arbitrary point in the object which is distance r from the axis and defined θ to be the angle the line that connects the axis and the point makes with x-axis. The angle keeps increasing as the object keeps rotating counterclockwise. Notice also that it doesn't matter which point we choose to denote the angle since if a point rotates 60 degrees, another point rotates 60 degrees as well. So, if the object rotates 60 degrees per second, we say that the angular velocity is 60 degrees per second. In other words, if the degree rotated is $\Delta\theta$ and the time it took to rotate that angle is Δt, the angular velocity ω (This Greek letter is pronounced as "omega") is given as follows:

$$\omega = \frac{\Delta\theta}{\Delta t} \tag{1}$$

In physics, it is customary to use radian instead of degree when denoting angle. Then, what is v the speed of a point in the object distance r from the axis? Recall our earlier article "Centripetal force." We had:

$$\Delta s = v\Delta t = r\Delta\theta \tag{2}$$

Therefore, we conclude:

$$v = r\frac{\Delta\theta}{\Delta t} = r\omega \tag{3}$$

Problem 1. If an object rotates twice a second at a constant angular velocity, what is its angular velocity in unit rad/s?

Problem 2. Suppose an object rotates with a constant angular velocity ω. How long does it take for it to complete one rotation? (Hint[1])

Problem 3. Suppose an object moves in a circular orbit with radius r and constant speed v. As the circumference of the circle is $2\pi r$, it will take

$$T = \frac{2\pi r}{v} \tag{4}$$

seconds to complete one rotation. This T is also known as "period." Express the above T in terms of ω using (3) and check that your answer is same as the answer of Problem 2.

[1] One rotation corresponds to 2π radians.

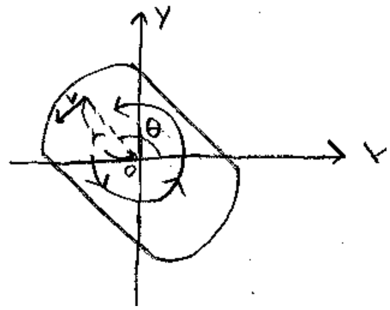

Figure 1: an object rotating

Figure 2: acceleration

Also, notice that angular velocity doesn't need to remain the same. It can increase or decrease if the object rotates faster and faster or slower and slower. For example, if an initial angular velocity was 10 rad/s, and if it increases with 2 rad/s per second, the final angular velocity after 3 seconds would be 16 rad/s. Therefore, if we denote the angular acceleration by α, (This Greek letter is pronounced as "alpha.") we have the following equation:

$$\omega_f = \omega_i + \alpha t \tag{5}$$

Or, we could write equivalently:

$$\alpha = \frac{\Delta\omega}{\Delta t} \tag{6}$$

If the angular acceleration is non-zero, angular velocity is either increasing or decreasing. However, from (3), it implies that the velocity rotating is either increasing or decreasing. So, how fast is it increasing or decreasing? This is something like acceleration. If we denote it by a_{tan} "tangential acceleration," (3) and (6) imply:

$$a_{tan} = \frac{\Delta v}{\Delta t} = r\frac{\Delta\omega}{\Delta t} = r\alpha \tag{7}$$

Notice that this is different from centripetal acceleration. See Fig. 2. Centripetal acceleration (denoted as $\vec{a}_c$ in the figure) is toward center, it is non-zero even when the angular acceleration is zero, as long as it is rotating at non-zero angular velocity. On the other hand, tangential acceleration is along tangential direction (i.e. along the direction the object moves) and is zero, when the angular acceleration is zero. In any case, the total acceleration is the vector sum of these two different components of acceleration.

Now, let's compare our formulas on rotation with the formulas on 1-dimensional motion explained in our earlier articles. As angular velocity, and angle are the rotational analog of velocity, and position, (1) is the rotational analog of following formula.

$$v = \frac{\Delta s}{\Delta t} \tag{8}$$

where Δs is the position changed (i.e. distance traveled) and Δt is the time it took.

Similarly, (5) is the rotational analog of the following formula.

$$v_f = v_i + at \tag{9}$$

As we learn that θ,ω, and α are the rotational analogs of s, v and a, the following equation

$$s = \frac{(v_i + v_f)t}{2} = v_i t + \frac{1}{2}at^2 \tag{10}$$

suggests the following equation.

$$\theta = \frac{(\omega_i + \omega_f)t}{2} = \omega_i t + \frac{1}{2}\alpha t^2 \tag{11}$$

In the next article, we will find what rotational analog of force is.

Problem 4. For a constant angular acceleration α, derive the following relation

$$\omega_f^2 = \omega_i^2 + 2\alpha\theta \tag{12}$$

where ω_i and ω_f are initial and final angular velocities and θ is the angle rotated during this interval. (Hint[2])

Summary

- In rotation, angular velocity is given by

$$\omega = \frac{\Delta\theta}{\Delta t}$$

- Angular acceleration is given by

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

  For a constant angular acceleration, we naturally have

$$\omega_f = \omega_i + \alpha t$$

  where ω_f is the final angular velocity, t seconds after the initial angular velocity ω_i.

- The speed of a point in the object distance r from the axis is given by

$$v = r\omega$$

[2] Remember how we derived a similar formula in "Constant acceleration in 1-dimension."