Rotation in the Cartesian coordinate system

Having introduced the trigonometric functions it might be appropriate to give an example of its application. We will consider how the Cartesian coordinate of a given point changes as it is rotated by a given angle θ counterclockwise. See Fig.1. You see that the initial point (x, y) becomes (x', y') as it is rotated by the angle θ. To obtain x' and y' in terms of x, y and θ, consider Fig.2. Here, the x axis is rotated by angle θ. The displacement of the point after the rotation in the direction of new x axis remains to be "x," as the x axis and the point (x, y) are rotated by same angle. Similarly, the displacement of the point after rotation in the direction perpendicular to the new x axis is "y." If you use this, you will easily see the followings:

$$x' = x\cos\theta - y\sin\theta \tag{1}$$
$$y' = x\sin\theta + y\cos\theta \tag{2}$$

Notice that the rotation must preserve the distance of the rotated point from the origin. This is natural, since you just rotate the point without making it farther or closer to the origin. Actually, this can be explicitly checked. Using Pythagoras' theorem, "$r(x, y)$" the distance from the origin to the point (x, y) is given by following formula.

$$r(x, y) = \sqrt{x^2 + y^2} \tag{3}$$

It is an easy exercise to check the following:

$$\begin{aligned} r(x', y') &= \sqrt{(x')^2 + (y')^2} = \sqrt{(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2} \\ &= \sqrt{x^2 + y^2} = r(x, y) \end{aligned} \tag{4}$$

Here, we used the following fact:

$$\cos^2\theta + \sin^2\theta = 1 \tag{5}$$

As an aside, in our later article "Lorentz transformation and Rotation, a comparison," we will talk about a rotation in spacetime instead of the one in space as we considered in this article.

Problem 1. Check (4)

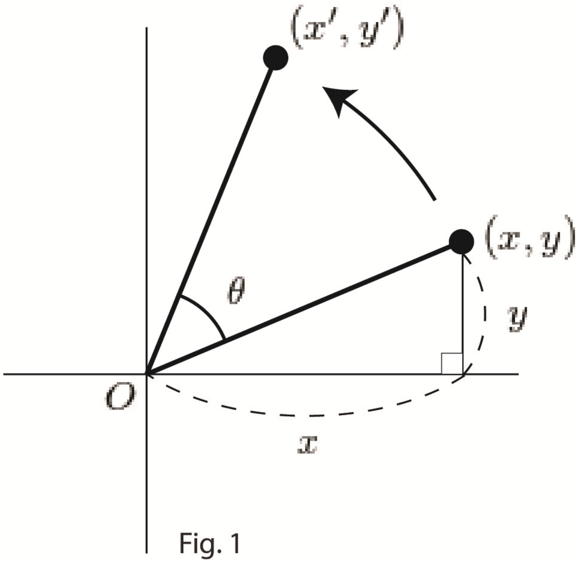

Fig. 1

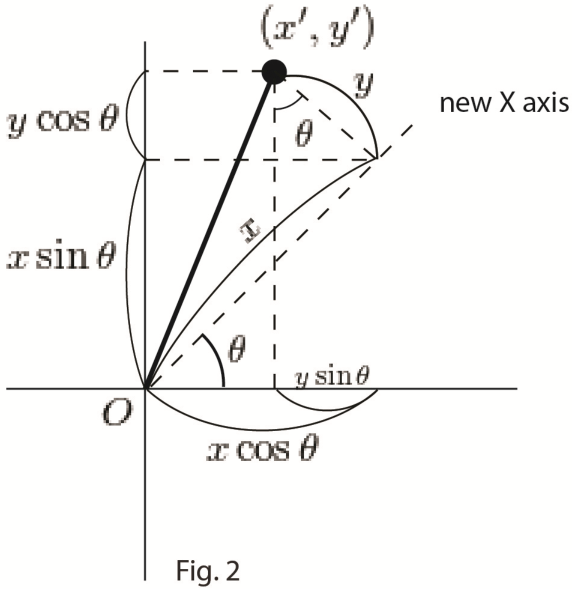

Fig. 2

Problem 2. How are (1) and (2) changed, if you rotate the point (x, y) clockwise instead of counterclockwise? (Hint[1])

Problem 3. Let's say that you rotate the point (x, y) counterclockwise by the angle θ, then rotate it again by the angle θ clockwise. Using the result of Problem 2, check that you come to the original point (x, y) as expected.

A comment. In this article, we have seen that the rotation doesn't change the distance between two points. (Well, we actually showed that the distance between the origin and a certain point doesn't change, but it can be easily generalized.) We have shown this by introducing a second coordinate system (x', y'). Another way of saying this is that the distance is same no matter in which coordinate system (i.e., either (x, y) system or (x', y') system) you measure the distance. Stepping further, I can make an interesting comment, if you know what a vector is. In a coordinate system (x, y), a vector may be expressed as

$$\vec{v} = v_x \hat{x} + v_y \hat{y} \tag{6}$$

However, in another coordinate system, the same vector may be expressed as

$$\vec{v} = v'_x \hat{x}' + v_y \hat{y}' \tag{7}$$

It goes without saying that the distance of $\vec{v}$, i.e., its magnitude

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{v_x'^2 + v_y'^2} \tag{8}$$

is same in the two coordinate systems.

[1] This is equivalent to rotate the point by $-\theta$ counter clockwise.

It may sound like a trivial remark, but the fact that components in all different coordinate systems satisfy the above relation has a far-reaching consequence than you now imagine.

Summary

- Under the rotation of a point with respect to the origin by the angle θ, the new coordinate of the point can be expressed by the original coordinates and $\cos\theta$ and $\sin\theta$.
- Under such a rotation, the distance to the origin doesn't change.