Separation of variables method in partial differential equations

In last article, we have seen how the PDE that describes a wave in 1-dimension and its solution looked like. Then, one may ask, what would the PDE that describes a wave in 3-dimension and its solution look like? The PDE is given by the following:

$$\frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \tag{1}$$

The strategy of solving this PDE is called "separation of variables" and goes as follows. Let's assume $h(x, y, z, t)$ can be expressed as follows:

$$h(x, y, z, t) = a(x)b(y)c(z)d(t) \tag{2}$$

Here, we apparently separated the variables. Now plug this into (1). Then, we get:

$$\begin{aligned}&\frac{1}{v^2}a(x)b(y)c(z)\frac{\partial^2 d(t)}{\partial t^2}\\ &\qquad = \frac{\partial^2 a(x)}{\partial x^2}b(y)c(z)d(t) + a(x)\frac{\partial^2 b(y)}{\partial y^2}c(z)d(t) + a(x)b(y)\frac{\partial^2 c(z)}{\partial z^2}d(t)\end{aligned}$$

Here, we see that the t derivatives act only on $d(t)$ and not on a, b, c as they don't involve t. The similar is true for other variables. For example, y derivatives act only on $b(y)$. If we divide the above equation by $abcd$ we get:

$$\frac{1}{v^2}\left(\frac{1}{d(t)}\frac{\partial^2 d(t)}{\partial t^2}\right) = \left(\frac{1}{a(x)}\frac{\partial^2 a(x)}{\partial x^2}\right) + \left(\frac{1}{b(y)}\frac{\partial^2 b(y)}{\partial y^2}\right) + \left(\frac{1}{c(z)}\frac{\partial^2 c(z)}{\partial z^2}\right) \tag{3}$$

Here, we see that each term in each parenthesis is a function of one variable only. This implies that the left-hand side, which is a function of t only is equal to the right-hand side which is the sum of a function of x only, a function of y only and a function of z only. Of course, something that depends on t only cannot be always equal to the sum of each term that depends on x, y, and z only, respectively. Therefore, they should not depend on the variables at all. Therefore, each term in the parenthesis must be constant that doesn't depend on any variables. If we let the constants be $-\omega^2$, $-k_x^2$, $-k_y^2$ and $-k_z^2$ respectively[1], we have:

$$\omega^2 = v^2(k_x^2 + k_y^2 + k_z^2) \tag{4}$$

and

$$\frac{1}{d(t)}\frac{\partial^2 d(t)}{\partial t^2} = -\omega^2 \tag{5}$$

$$\frac{1}{a(x)}\frac{\partial^2 a(x)}{\partial x^2} = -k_x^2 \tag{6}$$

$$\frac{1}{b(y)}\frac{\partial^2 b(y)}{\partial y^2} = -k_y^2 \tag{7}$$

$$\frac{1}{c(z)}\frac{\partial^2 c(z)}{\partial z^2} = -k_z^2 \tag{8}$$

The solutions are given by

$$d(t) = d_0 e^{\pm i\omega t} \tag{9}$$

$$a(x) = a_0 e^{\pm ik_x x} \tag{10}$$

$$b(y) = b_0 e^{\pm ik_y y} \tag{11}$$

$$c(z) = c_0 e^{\pm ik_z z} \tag{12}$$

Plugging this back to (2), we get:

$$h(x, y, z, t) = Ce^{i(\pm k_x x \pm k_y y \pm k_z z \pm \omega t)} \tag{13}$$

Now, using the fact that the sum of solutions is also a solution in our case, a general solution can be written as:

$$h(x, y, z, t) = \sum_{\omega, k_x, k_y, k_z} C_{\omega, k_x, k_y, k_z} e^{i(k_x x + k_y y + k_z z - \omega t)} \tag{14}$$

where (4) is satisfied. (Here, we simply chose the relative sign in front of ω to be negative, as we can choose any sign we want, since ks and ω can take both positive and negative values and we are summing over them, which means it doesn't matter at the end. This choice leads to the usual convention of travelling wave.) Here, we can easily see that h is a sinusoidal wave (i.e. a wave that is wiggly like a sine function) as the exponent is imaginary. Also, we can actually write the exponent as $i(\vec{k} \cdot \vec{x} - \omega t)$ if we let $\vec{k} = (k_x, k_y, k_z)$. This is a wave that travels in the direction $\vec{k}$, and the general solution (14) is the superposition of such waves as the summation symbol denotes.

[1] We chose the constants to be negative, because such cases are much more interesting than the cases with positive signs and lead to the solutions of sinusoidal wave. Of course, we could consider the cases with positive signs, but we don't in this article as the analysis is very similar to the cases with negative signs.

Actually, Maxwell derived (1) from his famous Maxwell equations that concern electromagnetic field, with v that coincides with the speed of light, thus showing that light is a electromagnetic wave. We will talk about this in our later article "light as electromagnetic waves."

Problem 1. In quantum mechanics, a partial differential equation called "Schrödinger equation" plays a central role. It is given by

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+V(x,y,z)\psi=E\psi \tag{15}$$

Show that we can apply the separation of variable method if V can be expressed as $V(x,y,z)=V_x(x)+V_y(y)+V_z(z)$.

Summary

- If the variables x,y,z in a partial differential equation can be separated in such a way that it can be put in the following form

  $$A(x)+B(y)+C(z)=0$$

  where $a(x)$ is an expression solely in terms of x, $B(y)$ solely in terms of y, and $C(z)$ solely in terms of z, we can set the each term $A(x)$, $B(y)$, $C(z)$ as a constant to solve the PDE.